Given: $\sigma_{xx} = 12$ ksi, $\sigma_{yy} = 6$ ksi, and $\tau_{xy} = 4$ ksi (cw)

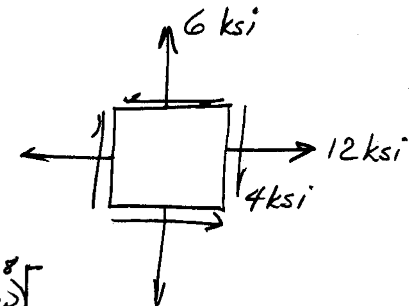
Find: (a) Use Mohr's circle to find principal normal stresses and max shear stress, and find angle between x-axis and σ_1.

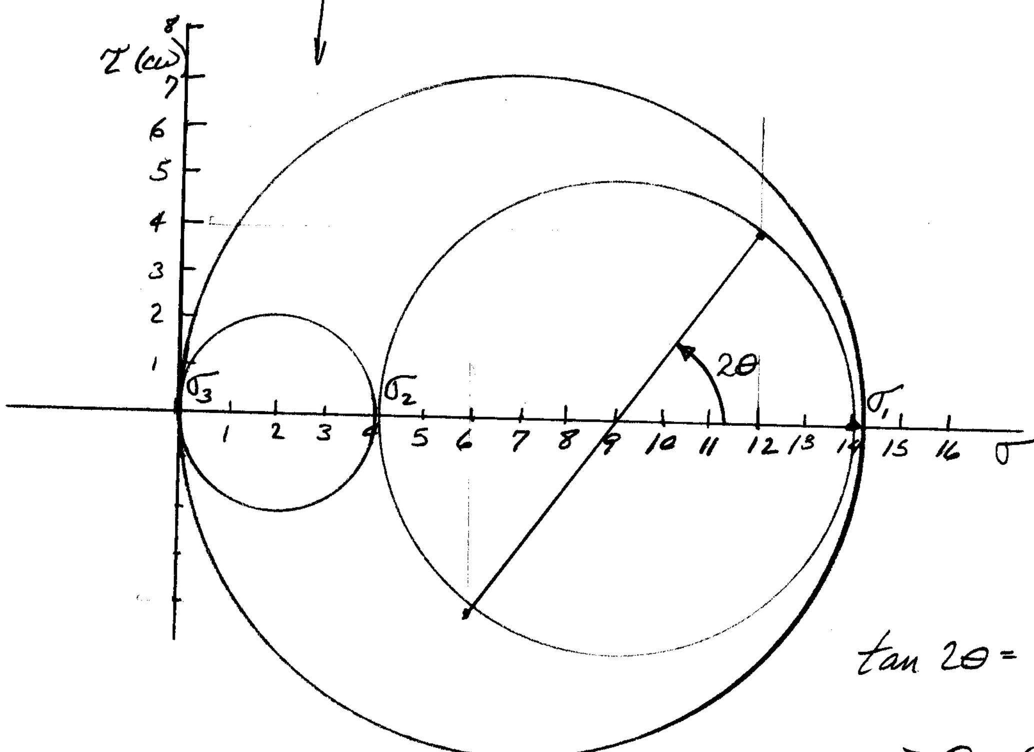
(b) Compute principal stresses using formulas.

(c) Cauchy stress tensor components. Find principal stresses using Matlab.

Solution:

(a)





$$\tan 2\theta = \frac{4}{3}$$

$$\Rightarrow \underline{\underline{\theta = 26.6^\circ}}$$

(b)

$$\sigma_1 = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{12+6}{2} + \sqrt{\left(\frac{12-6}{2}\right)^2 + 4^2} = \underline{\underline{14\text{ ksi}}}$$

$$\sigma_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{12+6}{2} - \sqrt{\left(\frac{12-6}{2}\right)^2 + 4^2} = \underline{\underline{4\text{ ksi}}}$$

$$\sigma_3 = 0$$

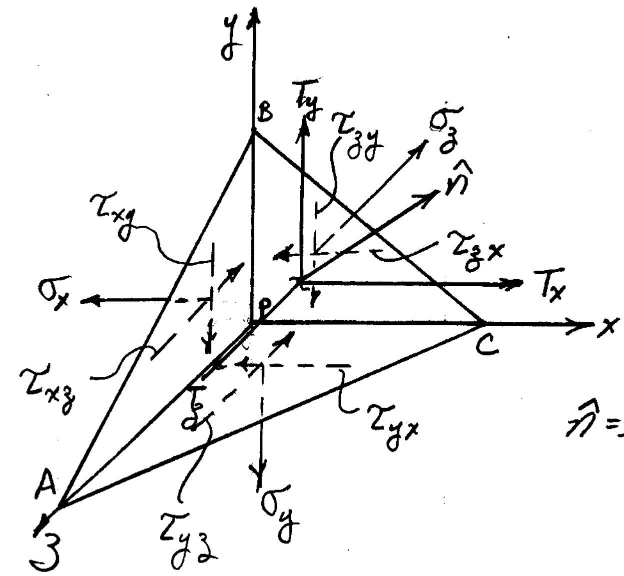
Note that these numbers agree with those obtained graphically using Mohr's circle.

(c)

$$\underset{\approx}{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 12 & -4 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stresses obtained using eig function in Matlab are $\sigma_1 = 14$ ksi; $\sigma_2 = 4$ ksi; $\sigma_3 = 0$

Given :



$$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$$

Find: Derive Cauchy's formula

Solution:

The area defined by the points A, B, and C may be written as

$$\vec{A} = A\hat{n} = A(l\hat{i} + m\hat{j} + n\hat{k})$$

The projection of the area in the x-direction can be found by taking the dot product with the unit vector $\hat{i}$.

$$A_x = \vec{A} \cdot \hat{i} = Al$$

A_x is the area defined by points A, P, and B.

Similarly, $A_y = \vec{A} \cdot \hat{j} = Am$ and

$$A_z = \vec{A} \cdot \hat{k} = An$$

where A_y is the area defined by A, P, and C, and A_z is the area defined by B, P, and C.

$$\sum F_x = 0 = -\sigma_x A_x - \tau_{yx} A_y - \tau_{zx} A_z + T_x A$$

$$\Rightarrow \sigma_x l \cancel{A} + \tau_{yx} m \cancel{A} + \tau_{zx} n \cancel{A} = T_x \cancel{A}$$

$$\sigma_x l + \tau_{yx} m + \tau_{zx} n = T_x$$

$$\sum F_y = 0 = -\sigma_y A_y - \tau_{xy} A_x - \tau_{zy} A_z + T_y A$$

$$\Rightarrow \tau_{xy} l \cancel{A} + \sigma_y m \cancel{A} + \tau_{zy} n \cancel{A} = T_y \cancel{A}$$

$$\tau_{xy} l + \sigma_y m + \tau_{zy} n = T_y$$

$$\sum F_z = 0 = -\tau_{xz} A_x - \tau_{yz} A_y - \sigma_z A_z + T_z A$$

$$\Rightarrow \tau_{xz} l \cancel{A} + \tau_{yz} m \cancel{A} + \sigma_z n \cancel{A} = T_z \cancel{A}$$

Combining the above three equations yields

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}$$

Note that the relationships

$\tau_{xy} = \tau_{yx}$, $\tau_{yx} = \tau_{xy}$, $\tau_{zy} = \tau_{yz}$, and $\tau_{zx} = \tau_{xz}$ were used.

Given: Cauchy stress tensor

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

where $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{zy} = \tau_{yz}$.

Find: The characteristic polynomial that can be solved to find the three principal stresses.

Solution: The characteristic equation is obtained from the eigenvalue problem

$$\det \begin{bmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{bmatrix} = 0$$

Expanding out the determinant yields

$$(\sigma_{xx} - \sigma)\left[(\sigma_{yy} - \sigma)(\sigma_{zz} - \sigma) - \tau_{yz}^2\right]$$

$$- \tau_{xy}\left[\tau_{yx}(\sigma_{zz} - \sigma) - \tau_{zx}\,\tau_{yz}\right]$$

$$+ \tau_{xz}\left[\tau_{yx}\,\tau_{zy} - \tau_{zx}(\sigma_{yy} - \sigma)\right] = 0$$

$$(\sigma_{xx}-\sigma)\left[\sigma_{yy}\sigma_{zz}-\sigma_{yy}\sigma-\sigma_{zz}\sigma+\sigma^2-\tau_{yz}^2\right]$$
$$-\tau_{xy}^2\sigma_{zz}+\tau_{xy}^2\sigma+\tau_{xy}\tau_{xz}\tau_{yz}$$
$$+\tau_{xz}\tau_{xy}\tau_{zy}-\tau_{xz}^2\sigma_{yy}+\tau_{xz}^2\sigma=0$$

$$\sigma_{xx}\cancel{\sigma_{yy}\sigma_{zz}}-\sigma_{xx}\cancel{\sigma_{yy}\sigma}-\sigma_{xx}\cancel{\sigma_{zz}}\sigma+\cancel{\sigma_{xx}\sigma^2}-\cancel{\tau_{yz}^2}\sigma_{xx}$$
$$-\sigma_{yy}\cancel{\sigma_{zz}\sigma}+\cancel{\sigma_{yy}}\sigma^2+\cancel{\sigma_{zz}\sigma^2}-\cancel{\sigma^3}+\cancel{\tau_{yz}^2\sigma}$$
$$-\tau_{xy}^2\cancel{\sigma_{zz}}+\cancel{\tau_{xy}^2\sigma}+\cancel{\tau_{xy}\tau_{xz}}\tau_{yz}$$
$$+\tau_{yz}\cancel{\tau_{xy}}\tau_{zy}-\tau_{xz}^2\sigma_{yy}+\cancel{\tau_{xz}^2}\sigma=0$$

Collecting terms yields,

$$-\sigma^3+(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})\sigma^2-(\sigma_{xx}\sigma_{yy}+\sigma_{xx}\sigma_{zz}+\sigma_{yy}\sigma_{zz}$$
$$-\tau_{yz}^2-\tau_{xy}^2-\tau_{xz}^2)\sigma+(\sigma_{xx}\sigma_{yy}\sigma_{zz}-\tau_{yz}^2\sigma_{xx}-\tau_{xy}^2\sigma_{zz}$$
$$+2\tau_{xy}\tau_{xz}\tau_{yz}-\tau_{xz}^2\sigma_{yy})=0$$

Dividing through by -1, yields eq. 3-10 in Shigley.

$$\sigma^3-(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})\sigma^2+(\sigma_{xx}\sigma_{yy}+\sigma_{xx}\sigma_{zz}+\sigma_{yy}\sigma_{zz}$$
$$-\tau_{yz}^2-\tau_{xy}^2-\tau_{xz}^2)\sigma-(\sigma_{xx}\sigma_{yy}\sigma_{zz}+2\tau_{xy}\tau_{xz}\tau_{yz}$$
$$-\tau_{yz}^2\sigma_{xx}-\tau_{xy}^2\sigma_{zz}-\tau_{xz}^2\sigma_{yy})=0$$