The University of Tennessee at Martin

MARTIN School of Engineering

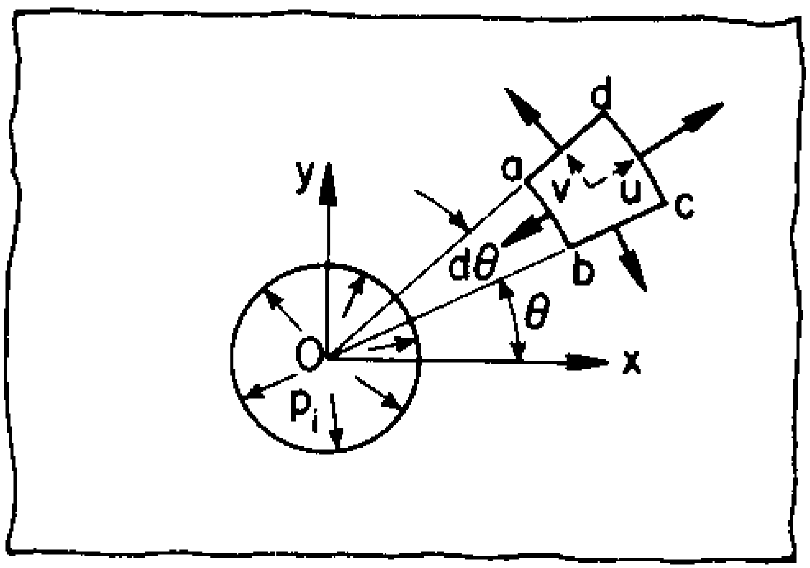
Thick Walled Cylinders

Lecture 13

Engineering 473
Machine Design

MARTIN

Axisymmetric Equation of Equilibrium
(Geometry)



$p_i \equiv$ internal pressure

$\theta \equiv$ angular position coordinate

$r \equiv$ radial position coordinate

$u \equiv$ displacement in r - direction

$v \equiv$ displacement in θ - direction

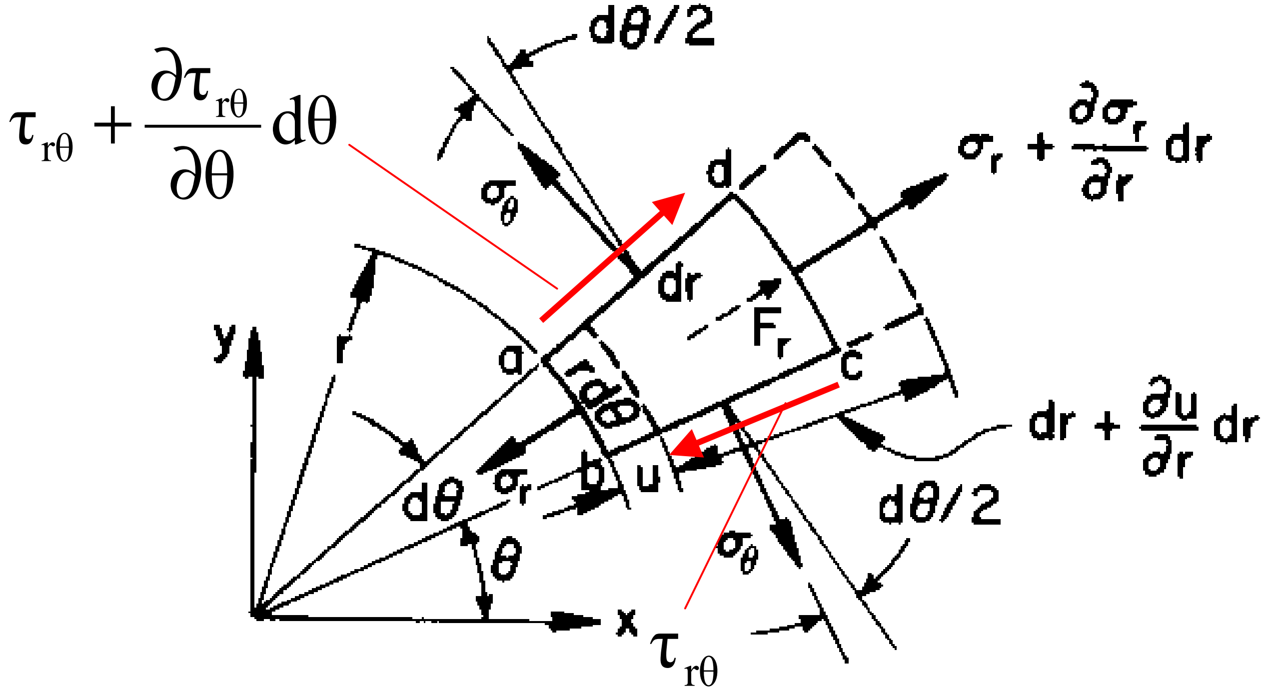
Axisymmetric $\equiv$ Nothing varies in the θ - direction.

$$\frac{\partial}{\partial \theta} = 0$$

Ugural, Fig. 8.1(a)

Axisymmetric Equation of Equilibrium

(Differential Element)



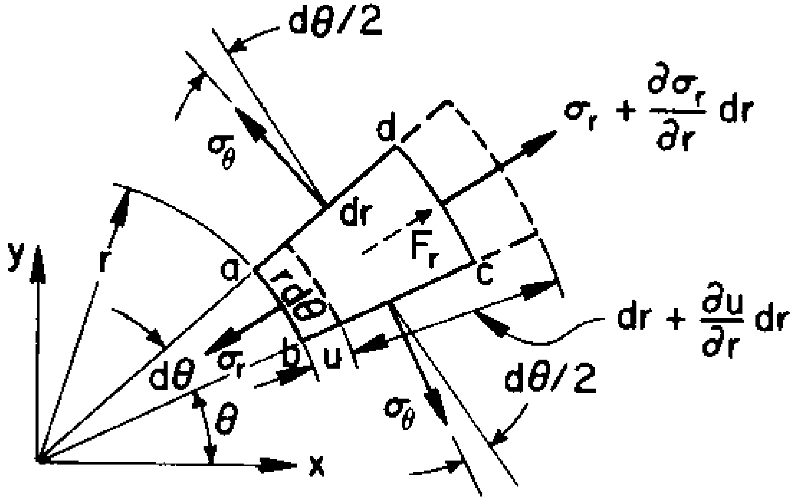
$$\frac{\partial \tau_{r\theta}}{\partial \theta} = 0, \text{ due to axisymmetric constraint}$$

$$\tau_{r\theta} = 0, \text{ due to stress compatibility}$$

Ugural, Fig. 8.1(b)

Axisymmetric Equation of Equilibrium

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right) \cdot (r + dr) d\theta \cdot dz - 2\sigma_\theta \sin\left(\frac{d\theta}{2}\right) drdz - \sigma_r \cdot rd\theta \cdot dz + F_r rd\theta \cdot dr \cdot dz = 0$$

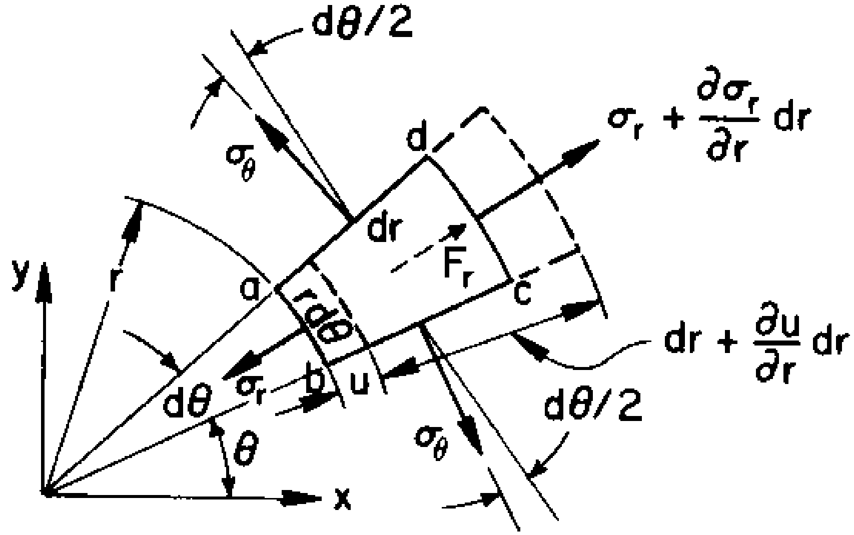


$$r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta + rF_r = 0$$

$$\boxed{\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0}$$

$F_r \equiv$ radial body force per unit volume

Strain Displacement Equations



$$\varepsilon_r = \frac{dr + \frac{\partial u}{\partial r}dr - dr}{dr} = \frac{du}{dr}$$

$$\varepsilon_\theta = \frac{(r+u)d\theta \text{ - } rd\theta}{rd\theta} = \frac{u}{r}$$

$$\varepsilon_r = \frac{du}{dr}, \ \varepsilon_\theta = \frac{u}{r}$$

Constitutive Equations

Hooke's Law

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right)$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right)$$

Stress-Strain equations are often referred to as **constitutive** equations, because they depend on what the part is made of. The equilibrium and strain-displacement equations are independent of the material.

Webster, "constitutive - making a thing what it is, essential"

Summary of Axisymmetric Equations

Equilibrium Equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

Strain-Displacement Equations

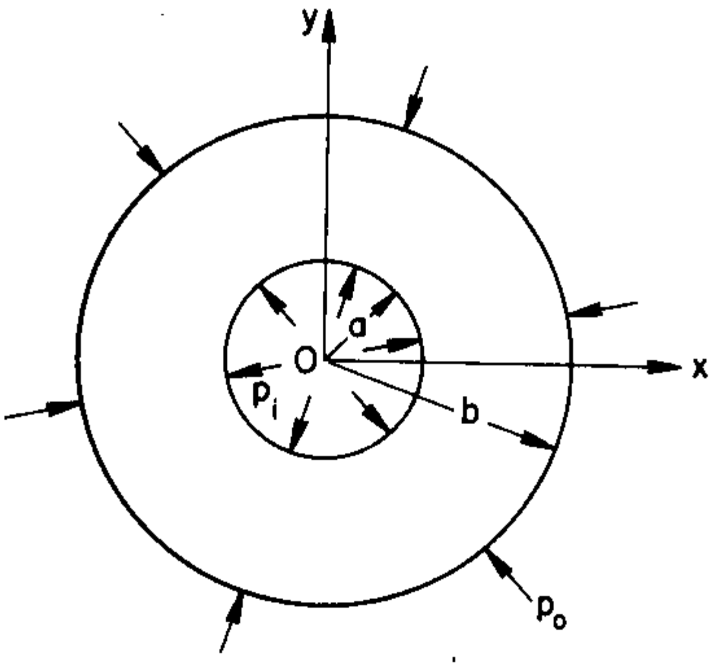
$$\varepsilon_r = \frac{du}{dr}, \ \varepsilon_\theta = \frac{u}{r}$$

Constitutive Equations

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right)$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

Thick Walled Cylinders
(Displacement Differential Equation)



$a \equiv$ inside radius

$b \equiv$ outside radius

$p_i \equiv$ internal pressure

$p_o \equiv$ external pressure

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

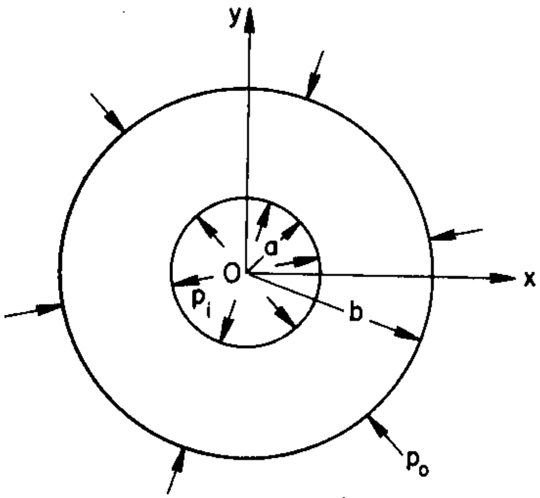
$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

Ugural, Fig. 8.2

Thick Walled Cylinders

(General Solution & Boundary Conditions)



$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

General Solution

$$u = C_1 r + \frac{C_2}{r}$$

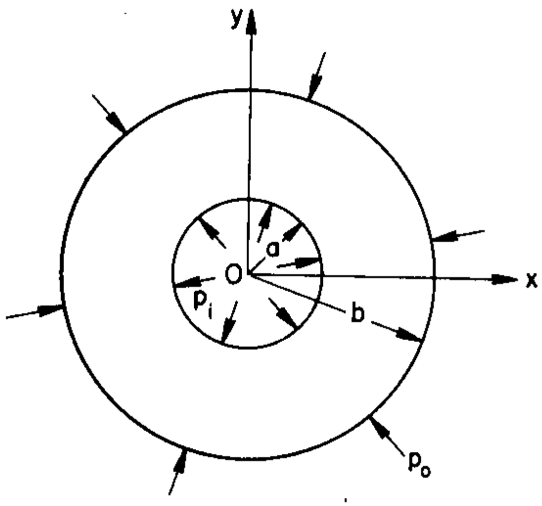
$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

Ugural, Fig. 8.2

Thick Walled Cylinders

(Boundary Conditions)



$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$-p_i = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{a^2}\right)\right]$$

$$-p_o = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{b^2}\right)\right]$$

Boundary Conditions

$$\sigma_r\big|_{r=a} = -p_i$$

$$\sigma_r\big|_{r=b} = -p_o$$

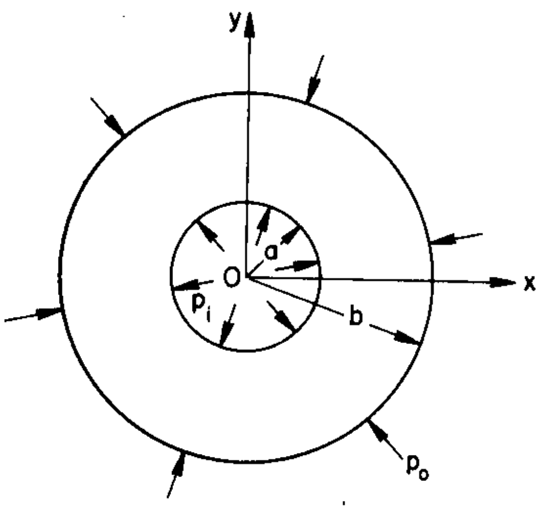
$$C_1 = \frac{1-\nu}{E}\left[\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}\right]$$

$$C_2 = \frac{1+\nu}{E}\left[\frac{a^2 b^2 (p_i - p_o)}{b^2 - a^2}\right]$$

Ugural, Fig. 8.2

Thick Walled Cylinders

(Lame' Equations)



$$\sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{(p_i - p_o) a^2 b^2}{(b^2 - a^2) r^2}$$

$$\sigma_\theta = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{(p_i - p_o) a^2 b^2}{(b^2 - a^2) r^2}$$

$$u = \frac{1-\nu}{E} \frac{(a^2 p_i - b^2 p_o) r}{b^2 - a^2} + \frac{1+\nu}{E} \frac{(p_i - p_o) a^2 b^2}{(b^2 - a^2) r}$$

Ugural, Fig. 8.2

Longitudinal Strain

(Unconstrained and Open Ends)

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_r - \nu\sigma_\theta\right)$$

Ends are unconstrained

and open, $\sigma_z = 0$

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_r + \sigma_\theta\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_r + \sigma_\theta = \frac{2E}{1-\nu^2}\left[C_1(1+\nu)\right]$$

$$\varepsilon_z = \frac{-2\nu \cdot C_1}{1-\nu}$$

$$\varepsilon_z = -\frac{2\nu}{E}\left(\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}\right)$$

Note that $\sigma_r + \sigma_\theta = \text{Constant}$

Longitudinal Stress

(Constrained Ends)

$$\varepsilon_z = 0 = \frac{1}{E}\left(\sigma_z - \nu\sigma_r - \nu\sigma_\theta\right)$$

$$\sigma_z = \nu\left(\sigma_r + \sigma_\theta\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

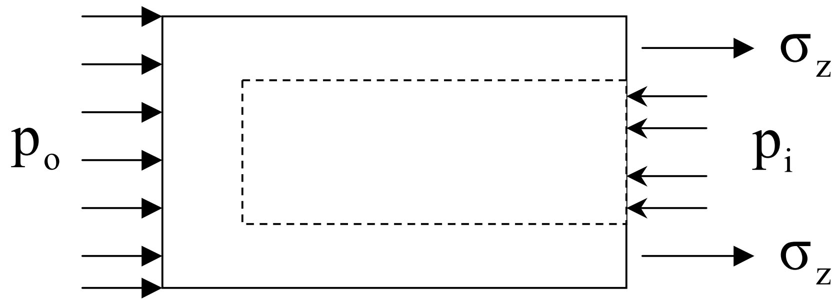
$$\sigma_r + \sigma_\theta = \frac{2E}{1-\nu^2}\left[C_1(1+\nu)\right]$$

$$\sigma_z = \nu\left(\frac{2EC_1}{1-\nu}\right)$$

$$\sigma_z = 2\nu\left(\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}\right)$$

Note that $\sigma_z = \text{Constant}$

Longitudinal Stress
(Closed and Unconstrained Ends)



$$\sigma_z \pi \left(b^2 - a^2\right) + p_o \pi \cdot b^2 - p_i \pi \cdot a^2 = 0$$

$$\sigma_z = \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

Special Cases

Internal Pressure Only

$$\sigma_r = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_z = 0, \text{ unconstrained}$$

$$\sigma_z = \frac{2\nu \cdot a^2 p_i}{b^2 - a^2}, \text{ constrained}$$

$$\sigma_z = \frac{a^2 p_i}{b^2 - a^2}, \text{ closed and unconstrained}$$

External Pressure Only

$$\sigma_r = -\frac{b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\frac{b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

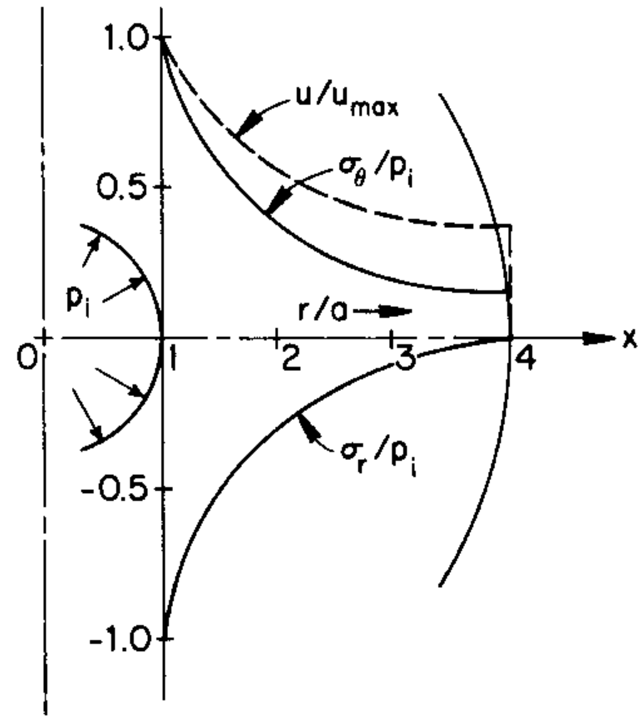
$$\sigma_z = 0, \text{ unconstrained}$$

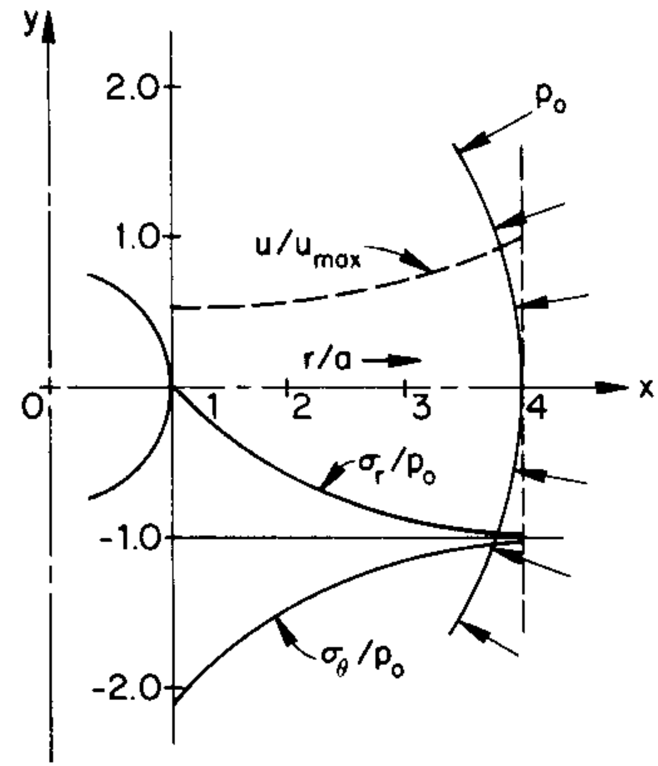
$$\sigma_z = -\frac{2\nu \cdot b^2 p_o}{b^2 - a^2}, \text{ constrained}$$

$$\sigma_z = -\frac{b^2 p_o}{b^2 - a^2}, \text{ closed \& unconstrained}$$

Stress Variation

b/a=4





Internal Pressure Only

External Pressure Only

Ugural, Fig. 8.3

Assignment

1. Show that the Lame' equations for the case of internal pressure reduce to the equations for a thin walled cylinder when the ratio b/a approaches 1.
2. A thick walled cylinder with 12 and 16 inch internal and external diameters is fabricated of a material whose tensile yield strength is 36 ksi and Poisson's ratio is 0.3. Calculate the von Mises stress when the internal pressure is 10 ksi. The cylinder has closed but unconstrained ends. Will the material yield?