The University of Tennessee at Martin
MARTIN School of Engineering

Stress at a Point

Lecture 3

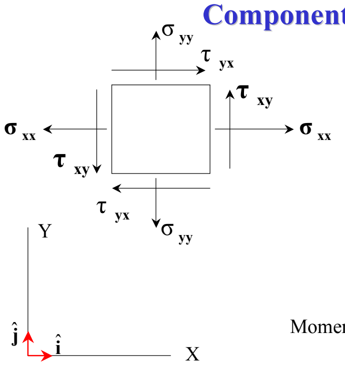
Engineering 473
Machine Design

Purpose

The stress state at critical locations in a machine component is required to evaluate whether the component will satisfy strength design requirements.

The purpose of this class is to review the concepts and equations used to evaluate the state of stress at a point.

2D Cartesian Stress Components



Notation

σ Normal Stress

τ Shear Stress

τ xy

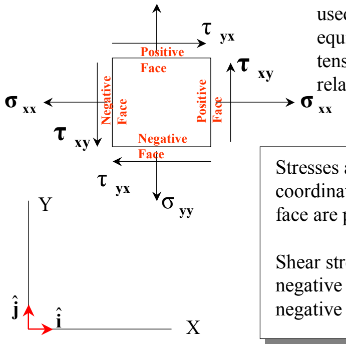
Face Direction

Moment equilibrium requires that

$$\tau_{xy} = \tau_{yx}$$

Tensor Sign Convention

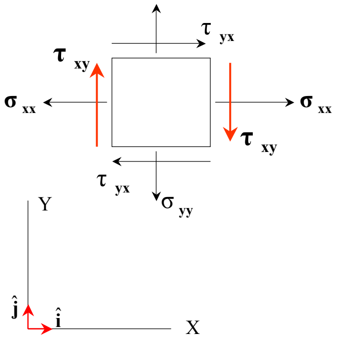
This sign convention must be used to satisfy the differential equilibrium equations and tensor transformation relationships.



Stresses acting in a positive coordinate direction on a positive face are positive.

Shear stresses acting in the negative coordinate direction on a negative face are positive.

2D Mohr's Circle Sign Convention

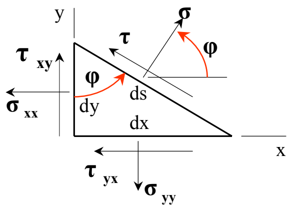


The sign convention used with the 2D Mohr's circle equations is slightly different.

A positive shear stress is one that tends to create clockwise (CW) rotation.

2D Mohr's Circle

(Transformation of Axis)



All equations for a 2-D Mohr's Circle are derived from this figure.

ΣF in the x- and y-directions yields the transformation-of-axis equations

$$\sigma = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos(2\varphi) + \tau_{xy}\sin(2\varphi)$$

$$\tau = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin(2\varphi) + \tau_{xy}\cos(2\varphi)$$

2D Mohr's Circle

(Principal Stress Equations)

The transformation-of-axis equations can be used to find planes for which the normal and shear stress are the largest.

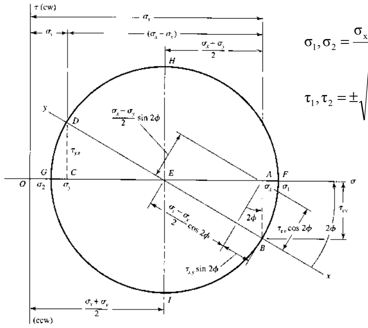
$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

We will use these equations extensively during this class.

2D Mohr's Circle

(Graphical Representation)



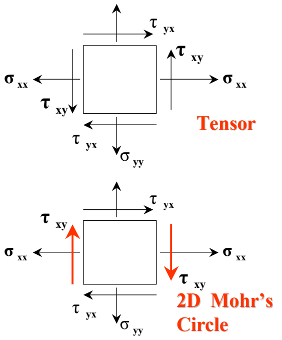
Shigley, Fig. 3.3

$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

Note that the shear stress acting on the plane associated with a principal stress is always zero.

Comments on Shear Stress Sign Convention



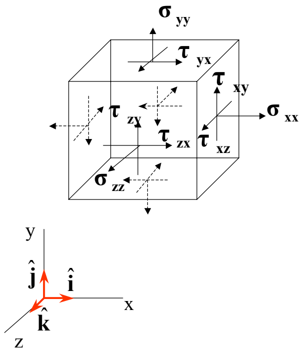
$$\sigma_1, \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

The sign convention is important when the transformation-of-axis equations are used.

The same answer is obtained when computing the principal stress components.

3D Stress Components



Note that the tensor sign convention is used.

There are nine components of stress.

Moment equilibrium can be used to reduce the number of stress components to six.

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

Cauchy Stress Tensor

Tensor Transformation Equation

$$\underset{\approx}{\sigma} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$\underset{\approx}{\sigma}$ is known as the Cauchy stress tensor. Its Cartesian components are shown written in matrix form.

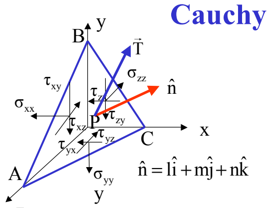
Tensors are quantities that are invariant to a coordinate transformation.

Tensor Transformation Equation

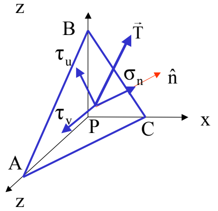
$$\sigma_{mn} = \beta_{mi}\,\sigma_{ij}\,\beta_{jn}$$

A vector is an example of a first order tensor. It can be written with respect to many different coordinate systems.

Cauchy Formula



$\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$



ΣF in the x,y,and z directions yields the Cauchy Stress Formula.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}$$

This equation is similar to the Mohr's circle transformation-of-axis equation

3D Principal Stresses

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix}$$

The shear stress on planes normal to the principal stress directions are zero.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \sigma \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

We need to find the plane in which the stress is in the direction of the outward unit normal.

$$\begin{bmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is a homogeneous linear equation.

3D Principal Stresses

(Eigenvalue Problem)

$$\begin{bmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

A homogeneous linear equation has a solution only if the determinant of the coefficient matrix is equal to zero.

$$\begin{vmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{vmatrix} = 0$$

This is an eigenvalue problem.

3D Principal Stresses

(Characteristic Equation)

$$\begin{vmatrix} (\sigma_{xx} - \sigma) & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & (\sigma_{yy} - \sigma) & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & (\sigma_{zz} - \sigma) \end{vmatrix} = 0$$

The determinant can be expanded to yield the equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

I_1, I_2, and I_3 are known as the first, second, and third invariants of the Cauchy stress tensor.

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

3D Principal Stresses

Characteristic Equation

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

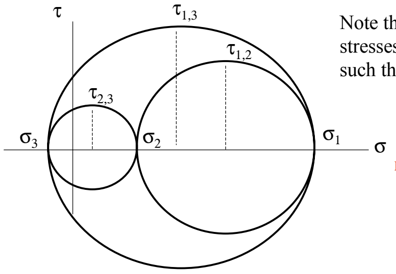
There are three roots to the characteristic equation, σ_1, σ_2, and σ_3.

Each root is one of the principal stresses.

The direction cosines define the principal directions or planes.

The direction cosines can be found by substituting the principal stresses into the homogeneous equation and solving.

3D Mohr's Circles



Note that the principal stresses have been ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

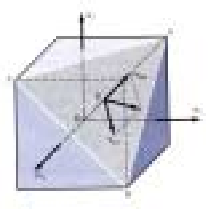
Maximum shear stresses

$$\tau_{1,2} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_{2,3} = \frac{\sigma_2 - \sigma_3}{2}$$

$$\tau_{1,3} = \frac{\sigma_1 - \sigma_3}{2}$$

Octahedral Stresses



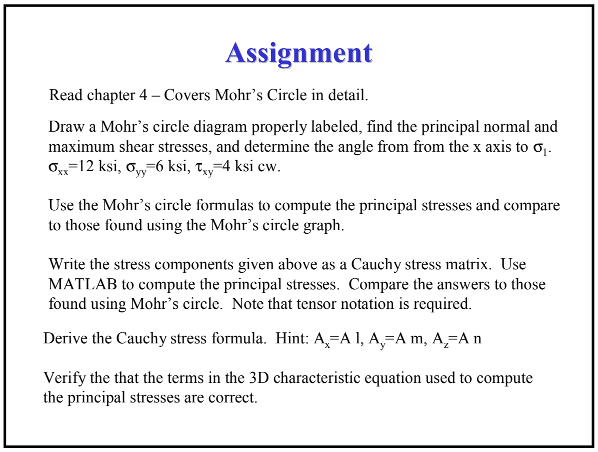
$$\sigma_{oct} = \frac{1}{3}I_1 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$\tau_{oct} = \frac{2}{3}\left(\tau_{1,2}^2 + \tau_{2,3}^2 + \tau_{1,3}^2\right)^{1/2}$$

$$= \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$= \frac{1}{3}\begin{bmatrix} (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 \\ + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right) \end{bmatrix}^{1/2}$$

Note that there eight corner planes in a cube.
Hence the name octahedral stress.



Assignment

Read chapter 4 – Covers Mohr's Circle in detail.

Draw a Mohr's circle diagram properly labeled, find the principal normal and maximum shear stresses, and determine the angle from from the x axis to σ_1. σ_{xx}=12 ksi, σ_{yy}=6 ksi, τ_{xy}=4 ksi cw.

Use the Mohr's circle formulas to compute the principal stresses and compare to those found using the Mohr's circle graph.

Write the stress components given above as a Cauchy stress matrix. Use MATLAB to compute the principal stresses. Compare the answers to those found using Mohr's circle. Note that tensor notation is required.

Derive the Cauchy stress formula. Hint: A_x=A l, A_y=A m, A_z=A n

Verify the that the terms in the 3D characteristic equation used to compute the principal stresses are correct.