The University of Tennessee at Martin
MARTIN School of Engineering

Steady Load Failure Theories
(Distortion Energy Theory)

Lecture 6

Engineering 473
Machine Design

Distortion-Energy Theory

Postulate: Yielding will occur when the distortion-energy per unit volume equals the distortion-energy per unit volume in a uniaxial tension specimen stressed to its yield strength.

Strain Energy

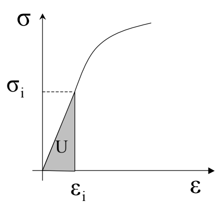
Strain Energy

The strain energy per unit volume is given by the equation

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

Units

$$[U] = \left[{}^{lb}\!/\!_{in^2}\right]\cdot\left[{}^{in}\!/\!_{in}\right] = \left[lb - {}^{in}\!/\!_{in^3}\right]$$



The strain energy in a tensile test specimen is the area under the stress-strain curve.

Strain Energy
(Elastic Stress-Strain Relationship)

An expression for the strain energy per unit volume in terms of stress only can be obtained by making use of the stress-strain relationship

Algebraic Format

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3)$$

$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

Matrix Format

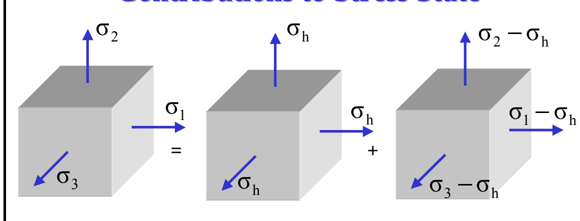
$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}$$

Strain Energy
(Stress Form of Equation)

$$\begin{aligned}U &= \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3\\ &= \frac{1}{2}\sigma_1\left\{\frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)\right\}\\ &+ \frac{1}{2}\sigma_2\left\{\frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3)\right\}\\ &+ \frac{1}{2}\sigma_3\left\{\frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)\right\}\end{aligned}$$

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

Distortion and Hydrostatic Contributions to Stress State



Principal Stresses Acting on Principal Planes

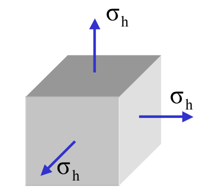
$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Hydrostatic Stress

Distortional Stresses

The distortional stress components are often called the deviatoric stress components.

Physical Significance
(Hydrostatic Component)



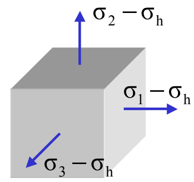
$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

The hydrostatic stress causes a change in the volume.

The cube gets bigger in tension, smaller in compression.

$$\sigma_h = Ke$$
$$K \equiv \text{Bulk Modulus}$$
$$e \equiv \text{volumetric strain}$$

Physical Significance
(Distortional Stresses)



These unequal stresses act to deform or distort the material element.

There is no change in volume, but there is a change in shape.

These stresses try to elongate or compress the material more in one direction than in another.

Strain Energy Associated with the Hydrostatic Stress

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$
$$U_h = \frac{1}{2E}\left[\sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu(\sigma_h\sigma_h + \sigma_h\sigma_h + \sigma_h\sigma_h)\right]$$
$$= \frac{1}{2E}\left[3\sigma_h^2 - 6\nu \cdot \sigma_h^2\right]$$
$$\boxed{U_h = \frac{3}{2}\frac{(1-2\nu)}{E}\sigma_h^2}$$

This term is equal to the strain energy per unit volume from the hydrostatic stress components.

Distortional Strain Energy

The distortional strain energy is equal to the difference between the total strain energy and the hydrostatic strain energy.

$$U_d = U - U_h$$
$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$
$$- \frac{3}{2}\frac{(1-2\nu)}{E}\frac{(\sigma_1 + \sigma_2 + \sigma_3)^2}{9}$$
$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$
$$- \frac{1}{2}\frac{(1-2\nu)}{3E}\begin{pmatrix} \sigma_1^2 + \sigma_1\sigma_2 + \sigma_1\sigma_3 \\ + \sigma_2^2 + \sigma_1\sigma_2 + \sigma_2\sigma_3 \\ + \sigma_3^2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \end{pmatrix}$$

Distortional Strain Energy
(Continued)

$$U_d = U - U_h$$
$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$
$$- \frac{1}{2}\frac{(1-2\nu)}{3E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right)$$
$$\boxed{U_d = \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]}$$

Distortional Strain Energy in Tension Test Specimen

Postulate: Yielding will occur when the distortion-energy per unit volume equals the distortion-energy per unit volume in a uniaxial tension specimen stressed to its yield strength.

$$U_d = \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]$$
$$U_d = \frac{1+\nu}{3E}S_y^2$$

Hamrock, Fig. 3.1

Distortion Energy Failure Theory

Equating the distortional strain energy at the point under consideration to the distortional strain energy in the tensile test specimen at the yield point yields

$$U_d = \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]$$

$$= \frac{1+\nu}{3E} S_y^2$$

$$S_y^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1$$

$$\sigma_{eff} = S_y$$

$$\sigma_{eff} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

Alternate Forms of Effective Stress

$$\sigma_{eff} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1} \qquad \text{Form 1}$$

$$\sigma_{eff} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \qquad \text{Form 2}$$

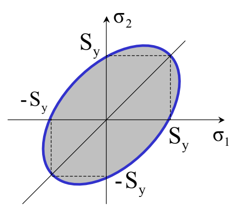
The effective stress is commonly referred to as the von Mises stress, after Dr. R. von Mises who contributed to the theory.

Plane Stress Condition

$$\sigma_3 = 0$$

$$\sigma_{eff} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$

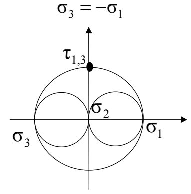
$$\sigma_{eff} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2}{2}}$$



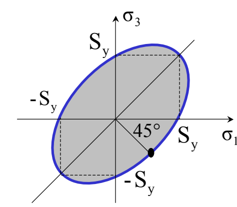
- As long as the stress state falls within the shaded area, the material will not yield.
- The surface, blue line, at which the material just begins to yield is called the yield surface.

Pure Shear Condition

$$\sigma_3 = -\sigma_1$$



Mohr's Circle for Pure Shear



$$\sigma_{eff} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3}$$

$$= \sqrt{3\sigma_1^2} = \sqrt{3\tau_{max}^2} = S_y$$

This is an important result.

$$\tau_{max} = 0.577 \cdot S_y = S_{ys}$$

Yield Surface in 3-D Stress State

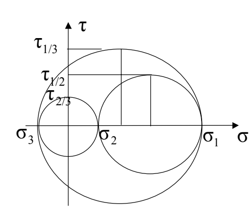
Hamrock, Fig. 6.9

Other Names for Distortion Energy Theory

$$\sigma_{eff} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

People came up with the same equation using different starting points.

- Shear Energy Theory
- Von Mises-Hencky Theory
- Octahedral-Shear-Stress Theory



$$\sigma_1 > \sigma_2 > \sigma_3$$

Assignment

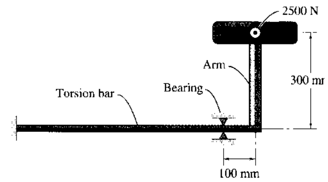
- Show that the two forms of the equation for the effective stress are equal.
- Show that the effective stress for a hydrostatic stress state is zero.
- Compute the effective stress at the critical location in the stepped shaft loaded in tension (previous assignment). The yield strength of the material is 30 ksi. Will the material yield at the critical location?

$$\sigma_{eff} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1}$$

$$\sigma_{eff} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

Assignment
(Continued)

In the rear wheel suspension of the Volkswagen "Beetle" the spring motion was provided by a torsion bar fastened to an arm on which the wheel was mounted. See the figure for more details. The torque in the torsion bar was created by a 2500-N force acting on the wheel from the ground through a 300-mm lever arm. Because of space limitations, the bearing holding the torsion bar was situated 100-mm from the wheel shaft. The diameter of the torsion bar was 28-mm. Find the von Mises stress in the torsion bar at the bearing.



Hamrock, Fig. 6.12