There are 14 problems on this exam. Carefully read and follow all directions. In order to receive credit show all necessary work. No credit will be given for an answer I cannot find or cannot read. Unless specified otherwise, all answers should be exact. Each problem is worth 6 points unless indicated otherwise.

Let \( f(x, y) = 4x^2 + 4xy + 68x + 6y^2 + 64y \). Use this function to work problems 1-8. Simplify all of your answers.

1. Determine the directional derivative of \( f \) in the direction \( \langle 4, -2 \rangle \) at the point \( (0, 1) \).
   \[
   \nabla f = \langle 8x + 4y + 68, 4x + 12y + 64 \rangle
   \]
   \[
   \nabla f \cdot \vec{u} = \langle 72, 76 \rangle \cdot \langle \frac{4}{\sqrt{20}}, -\frac{2}{\sqrt{20}} \rangle = \frac{136}{\sqrt{20}} = \frac{68}{\sqrt{5}}
   \]

2. In what direction should we go from the point \( (0, 1) \) if we want the maximal decrease in \( z = f(x, y) \)? (3 points)
   \[
   \vec{u} = \frac{-1}{\sqrt{10760}} \langle 72, 76 \rangle
   \]

3. Determine the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \( (0, 1, 70) \).
   \[
   72(x-0) + 76(y-1) - (z-70) = 0
   \]

4. Determine the equation of the normal line to the surface \( z = f(x, y) \) at the point \( (0, 1, 70) \).
   \[
   \langle 0, 1, 70 \rangle + t \langle 72, 76, -1 \rangle
   \]
5. Determine the critical point(s) for \( f(x, y) = 4x^2 + 4xy + 68x + 6y^2 + 64y \) and identify each point as a local maximum, a local minimum, or a saddle point.

\[
\begin{align*}
8x + 4y + 68 &= 0 \\
4x + 12y + 64 &= 0 \\
-20y - 60 &= 0 \\
y &= -3 \\
x &= -7 \\
\end{align*}
\]

Critical point \((-7, -3)\)

This is a local min

\[
\begin{align*}
x' &= 8 \\
y' &= 4 \\
x'' &= 12 \\
D &= 8 \cdot 12 - 4^2 = 80
\end{align*}
\]

6. Determine the maximum and minimum values of \( f(x, y) \) if \((x, y)\) is a point on or inside the triangle with vertices \((-8, -4), (4, -4), \) and \((-8, 8)\). The corner points and the critical point on the horizontal leg of the triangle have been filled in in the table below. Complete the table and then identify the maximum and minimum values for \( f \). (15 points)

<table>
<thead>
<tr>
<th>Points to Check</th>
<th>( f(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-8, -4))</td>
<td>-320</td>
</tr>
<tr>
<td>((4, -4))</td>
<td>112</td>
</tr>
<tr>
<td>((-8, 8))</td>
<td>352 (\text{max})</td>
</tr>
<tr>
<td>((-\frac{13}{2}, -4))</td>
<td>-329</td>
</tr>
<tr>
<td>((-8, -\frac{8}{3}))</td>
<td>-(\frac{992}{3})</td>
</tr>
<tr>
<td>((-\frac{1}{3}, \frac{1}{3}))</td>
<td>-(\frac{2}{3})</td>
</tr>
<tr>
<td>((-7, -3))</td>
<td>-334 (\text{min})</td>
</tr>
</tbody>
</table>

***Vertical Leg***

\(x = -8\),

\(-4 \leq y \leq 8\)

\(L = 6y^2 + 32y - 288\)

\(L' = 12y + 32\)

\(y = -\frac{8}{3}\)

***Hypotenuse***

\(y = -x\),

\(-8 \leq x \leq 4\)

\(L = 6x^2 + 4x\)

\(L' = 12x + 4\)

\(x = -\frac{1}{3}\)
7. Use differentials to approximate the value of \( f(0.02, 0.92) \).

\[
\Delta x = dx = 0.02 \quad \Delta y = dy = -0.08
\]

\[
\Delta z \approx dz = 72 \cdot 0.02 + 76 \cdot (-0.08) = -4.64
\]

\( f(0.02, 0.92) \approx 65.36 \)

8. Let \( x = r^2 s^3 \) and \( y = e^{rs^2} \). Determine \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial s} \). (12 points)

\[
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}
\]

\[
= (8x + 4y + 68) \cdot 2rs^3 + (4x + 12y + 64) \cdot \frac{2}{r} s^2
\]

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}
\]

\[
= (8x + 4y + 68) \cdot 3r^2 s^2 + (4x + 12y + 64) \cdot 2rse^{rs^2}
\]
Use the functions given below to work problems 9-14.

\[ a(x, y) = -3x \cos(xy) \quad b(x, y) = \frac{\sin(2x^2 + y^2 - 1)}{4x^2 + y^2 - 1} \]
\[ c(x, y) = \frac{x + 7y + 7}{\sqrt{-x - 7y + 7}} \quad d(x, y) = \ln(-x^2 - 2y + 1) \]

9. Draw the region in the xy-plane that is the domain of \( d(x, y) \).

\[-x^2 - 2y + 1 > 0 \]
\[ y < \frac{1}{2} x^2 + \frac{1}{2} \]

10. Determine the two first order partial derivatives and the four second order partial derivatives for \( a(x, y) \). (12 points)

\[ \frac{\partial a}{\partial x} = -3 \cos(xy) + 3xy \sin(xy) \]
\[ \frac{\partial a}{\partial y} = 3x^2 \sin(xy) \]
\[ \frac{\partial^2 a}{\partial x^2} = 6y \sin(xy) + 3x^2 \cos(xy) \]
\[ \frac{\partial^2 a}{\partial y^2} = 3x^3 \cos(xy) \]
\[ \frac{\partial^2 a}{\partial x \partial y} = \frac{\partial^2 a}{\partial y \partial x} = 6x \sin(xy) + 3x^2 \cos(xy) \]
11. Determine the slope of the tangent line to the graph of the curve that is the intersection of the surface $z = a(x, y)$ and the plane $x = 1$ at the point $(1, \pi, 3)$. Determine the vector equation of this tangent line.

\[
\text{slope}: \left. \frac{\partial y}{\partial x} \right|_{(1, \pi)} = 0
\]

\[
\langle 1, \pi, 3 \rangle + t \langle 0, 1, 0 \rangle
\]

12. Determine the slope of the tangent line to the graph of the curve that is the intersection of the surface $z = a(x, y)$ and the plane $y = \pi$ at the point $(1, \pi, 3)$. Determine the vector equation of this tangent line.

\[
\text{slope}: \left. \frac{\partial y}{\partial x} \right|_{(1, \pi)} = 3
\]

\[
\langle 1, \pi, 3 \rangle + t \langle 1, 0, 3 \rangle
\]

13. Determine the equation of the plane that contains the two lines from problems 11 and 12.

\[
\begin{vmatrix}
i & j & k \\
0 & 1 & 0 \\
1 & 0 & 3 \\
\end{vmatrix} = \langle 3, 0, -1 \rangle
\]

\[3(x-1)+0(y-\pi)-(z-3)=0\]

14. Evaluate each of the following limits when they exist. If the limit does not exist, explain why not. (3 points each)

(a) \(\lim_{(x,y) \to (-2,1)} d(x,y)\) = \text{DNE}

\(-x^2 - 2y + 1\) is negative

(b) \(\lim_{(x,y) \to (0,0)} a(x,y)\) = 0