Review for Math 320, Exam III

There are more problems on this review than I can ask on a 50-minute exam. I will choose enough problems to get a 100-point exam. Each problem is worth 6 points unless indicated otherwise.

Use the function \( f(x, y) = xy(\cos y^2) + 2x^3y^2 - 4y \) in problems 1-5.

1. Determine \( \frac{\partial f}{\partial x} \).

\[
y \cos(y^2) + 6x^2y^2
\]

2. Determine \( \frac{\partial f}{\partial y} \).

\[
x \cos(y^2) - 2xy^2 \sin(y^2) + 4x^3y - 4
\]

3. Determine \( \frac{\partial^2 f}{\partial x \partial y} \).

\[
\cos(y^2) - 2y^2 \sin(y^2) + 12x^2y
\]

4. Determine the equation of the tangent line to the curve that is the intersection of the surface \( z = xy(\cos y^2) + 2x^3y^2 - 4y \) with the plane \( x = 3 \) at the point \( (3, 0, 0) \).

At \( (3,0,0) \), \( \frac{\partial f}{\partial y} = -1 \)

\[
\langle 3, 0, 0 \rangle + t \langle 0, 1, -1 \rangle
\]

5. Determine \( \lim_{(x,y) \to (2,\sqrt{\pi})} f(x, y) \).

\[
= 2\sqrt{\pi} \cos(\sqrt{\pi}) + 16\pi - 4\sqrt{\pi}
\]

\[
= 16\pi - 6\sqrt{\pi}
\]
6. Sketch the graph of the domain of the function \( G(x, y) = \sqrt{2-y-3x} \).

\[ 2y - 3x > 0 \]

Let \( F(x, y) = 2x^2 + 2xy + y^2 + 2y \). Use this function for problems 7-14.

7. Determine the directional derivative of \( F \) in the direction of the vector \( \langle 3, -4 \rangle \) at the point \( (2, 5) \).

\[ \nabla F = \langle 4x + 2y, 2x + 2y + 2 \rangle = \langle 18, 16 \rangle \text{ at } (2, 5) \]

\[ \langle 18, 16 \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{-10}{5} = -2 \]

8. In what direction should we go from the point \( (2, 5) \) if we want the maximal increase in \( z = F(x, y) \)?

\[ \frac{1}{\sqrt{18^2 + 16^2}} \langle 18, 16 \rangle = \frac{1}{\sqrt{580}} \langle 18, 16 \rangle \]

9. Determine the equation of the tangent plane to the surface \( z = F(x, y) \) at the point \( (2, 5, 63) \).

\[ 18(x - 2) + 16(y - 5) - (z - 63) = 0 \]

10. Determine the equation of the normal line to the surface \( z = F(x, y) \) at the point \( (2, 5, 63) \).

\[ \langle 2, 5, 63 \rangle + t \langle 18, 16, -1 \rangle \]
11. Determine and identify all local extreme points for \( F(x, y) = 2x^2 + 2xy + y^2 + 2y \).

\[
\begin{align*}
\text{Solve } & \quad 4x + 2y = 0 \\
& \quad 2x + 2y + 2 = 0 \\
\frac{\partial^2 F}{\partial x^2} &= 4 \\
\frac{\partial^2 F}{\partial x \partial y} &= 2 \\
\frac{\partial^2 F}{\partial y^2} &= 2 \\
x &= 1 \\
y &= -2 \\
D &= 4 \cdot 2 - 2^2 = 4 \\
(1, -2, -2) \text{ is local minimum}
\end{align*}
\]

12. Determine the maximum and minimum values of \( F(x, y) \) if \((x, y)\) is a point on or inside the triangle with vertices \((4, 0), (0, 6), \) and \((4, 6)\). (18 points)

Along \( y = 6 \); \( 0 \leq x \leq 4 \)

\[
\begin{align*}
F(x, 6) &= 2x^2 + 12x + 48 \\
F'(x) &= 4x + 12 \\
F(0, 6) &= 48 \\
F(4, 6) &= 128
\end{align*}
\]

Along \( x = 4 \); \( 0 \leq y \leq 6 \)

\[
\begin{align*}
F(4, y) &= y^2 + 10y + 32 \\
F'(y) &= 2y + 10 \\
F(4, 0) &= 32 \\
F(4, 6) &= 128
\end{align*}
\]

128 is max

31.8 is min

Along \( y = -\frac{3}{2}x + 6 \);

\[
\begin{align*}
F(x, y) &= 2x^2 + 2x(-\frac{3}{2}x + 6) \\
&\quad + (-\frac{3}{2}x + 6)^2 + 2(-\frac{3}{2}x + 6) \\
&= \frac{5}{4}x^2 - 9x + 48 \\
F'(x) &= \frac{5}{2}x - 9 \\
F(0, 6) &= 48 \\
F(3.6, 0.6) &= 31.8 \\
F(4, 0) &= 32
\end{align*}
\]
13. Use differentials to approximate the value of \( F(2.02, 4.95) \).

\[
F(2.02, 4.95) \approx 18(2.02 - 2) + 16(4.95 - 5) + 63
162.56
\]

14. Use the chain rule to determine \( \frac{\partial F}{\partial r} \) and \( \frac{\partial F}{\partial s} \) if \( x = 2r^2 s^3 \) and \( y = \cos(rs) \). (10 points)

\[
\frac{\partial F}{\partial r} = (4x + 2y) \cdot 4rs^3 + (2x + 2y + 2)(-s \sin(rs))
\]

\[
\frac{\partial F}{\partial s} = (4x + 2y) \cdot 6r^2 s^2 + (2x + 2y + 2)(-r \sin(rs))
\]
15. Use Lagrange multipliers to determine the maximum and minimum values of
\( g(x, y) = xy \) subject to the constraint \( x^2 + y^2 = 4 \). (18 points)

\[
\begin{align*}
  y &= \lambda (2x) \\
  x &= \lambda (2y)
\end{align*}
\]

\( x = 0 \Rightarrow y = 0 \) not possible

\[
\lambda = \frac{y}{2x}
\]

\[
\begin{align*}
  x &= \frac{y}{2x} \cdot 2y \\
  x^2 &= y^2 \\
  2y^2 &= 4 \\
  y^2 &= 2 \\
  y &= \pm \sqrt{2}
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
\lambda & x & y & xy \\
\hline
\frac{1}{2} & \sqrt{2} & \sqrt{2} & 2 \text{ max} \\
-\frac{1}{2} & -\sqrt{2} & \sqrt{2} & -2 \text{ min} \\
-\frac{1}{2} & \sqrt{2} & -\sqrt{2} & -2 \text{ min} \\
\frac{1}{2} & -\sqrt{2} & -\sqrt{2} & 2 \text{ max}
\end{array}
\]