MATH 330. TEST 1. (HARVEY FALL 2005)

Name: (2 points) ____

No notes or texts allowed. Show all work.

1 (6 points). Consider the differential equation:

$$\frac{dy}{dx} = x + y$$

Sketch a direction field (put a direction arrow on the nine points $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$.
Now sketch the solution curve $y(x)$ which matches the initial value $y(0) = 0$.

2 (6 points). (a) Identify the following differential equations as linear or nonlinear; (b) Give the order of each differential equation:

$$(1)\quad \frac{d^3y}{dx^3} + x \cdot y^2 = 4x - 5 \qquad\qquad (2)\quad \frac{d^2y}{dx^2} + 4x \cdot \frac{dy}{dx} + 16y = 2x^2 - 1$$

3 (6 points). Find all the values of c for which $f(x) = c \cdot x^{-1}$ a solution to the differential equation

$$y' + y^2 = 0$$

4-10 (10 points each). Solve the differential equations. Unless it is convenient to solve for an explicit solution, an implicit equation is sufficient.

4.

$$\frac{dy}{dx} = \frac{x^3}{1 - y^2}$$

5.

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

6.

$$x\frac{dy}{dx} + 4y = 1 + x^3$$

7.

$$(2x + y)\, dx + (x + 3y^2)\, dy = 0$$

8.

$$\frac{dy}{dx} - 2xy = 2x^3$$

9.

$$y^2\,dx + (2xy - e^{-2y})\,dy = 0$$

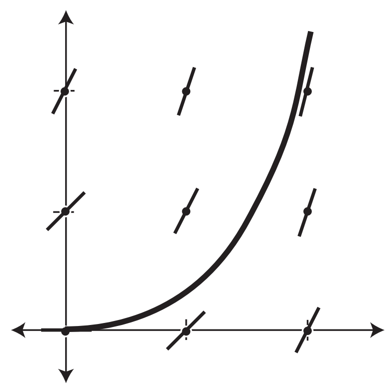
10.

$$\frac{dy}{dx} = \frac{x(1+y^2)}{1+x^2}$$

11 (12 points). A 50 gallon tank is filled with a 2% acid solution. The acidity will be increased by mixing in a 10% acid solution at a rate of 2 gal/min. Excess will be drained off into a storage container (which is initially empty). How long until the tank reaches 5% acidity? What will the (percent) acidity of the storage container be at that point?

Solutions

1.



2. (1) nonlinear, order 3. (2) linear, order 2

3.

$$f'(x) = -cx^{-2} \implies y' + y^2 = \frac{-c}{x^2} + \left(\frac{c}{x}\right)^2$$

$$y' + y^2 = 0 \implies c^2 - c = 0 \implies c = 0 \text{ or } 1$$

4.

$$(1-y^2)\,dy = x^3\,dx \implies y - \frac{y^3}{3} = \frac{x^4}{4} + C$$

5.

$$\frac{dy}{dx} = \frac{4x^2+3y^2}{2xy} = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)$$

Substitute:

$$v = y/x \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{2}{v} + \frac{3}{2}v \implies x\frac{dv}{dx} = \frac{2}{v} + \frac{1}{2}v \implies x\frac{dv}{dx} = \frac{4+v^2}{2v}$$

$$\frac{2v}{4+v^2}\,dv = \frac{dx}{x} \implies \ln(4+v^2) = \ln(x) + k \implies 4+v^2 = e^k x$$

$$4 + (y/x)^2 = Cx \implies y^2 = Cx^3 - 4x^2$$

6.

$$\frac{dy}{dx} + \frac{4}{x}y = \frac{1}{x} + x^2$$

Let

$$\mu = \exp\left(\int \frac{4}{x}\,dx\right) = x^4$$

$$x^4\frac{dy}{dx} + 4x^3y = x^3 + x^6 \implies \frac{d}{dx}(x^4y) = x^3 + x^6$$

$$\implies x^4y = \frac{x^4}{4} + \frac{x^7}{7} + C \implies y = \frac{1}{4} + \frac{x^3}{7} + \frac{C}{x^4}$$

7.

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

This equation is exact.

$$\frac{\partial F}{\partial x} = 2x + y \implies F = x^2 + xy + g(y)$$

$$\frac{\partial F}{\partial y} = x + g'(y) = x + 3y^2 \implies g'(y) = 3y^2 \implies g(y) = y^3$$

$$F = x^2 + xy + y^3 \qquad \implies \text{sol: } x^2 + xy + y^3 = C$$

8.

$$\mu = \exp\left(\int 2x\,dx\right) = e^{x^2}$$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2x^3e^{x^2} \implies \frac{d}{dx}(e^{x^2}y) = 2x^3e^{x^2}$$

Use the u-substitution $u = x^2$, $du = 2x\,dx$ to solve the integral on the right:

$$\int 2x^3e^{x^2}\,dx = \int ue^u\,du = ue^u - e^u + C = x^2e^{x^2} - e^{x^2} + C$$

$$\implies e^{x^2}y = x^2e^{x^2} - e^{x^2} + C \implies y = x - 1 + Ce^{-x^2}$$

9.

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$$

This equation is exact.

$$\frac{\partial F}{\partial x} = y^2 \implies F = xy^2 + g(y)$$

$$\frac{\partial F}{\partial y} = 2xy + g'(y) = 2xy - e^{-2y} \implies g'(y) = -e^{-2y} \implies g(y) = \frac{e^{-2y}}{2}$$

$$F = xy^2 + \frac{e^{-2y}}{2} \qquad \Longrightarrow \text{sol: } xy^2 + \frac{e^{-2y}}{2} = C$$

10.

$$\frac{dy}{1+y^2} = \frac{x}{1+x^2}\,dx \implies \tan^{-1}(y) = \frac{1}{2}\ln(1+x^2) + C \implies y = \tan\left(\frac{1}{2}\ln(1+x^2) + C\right)$$

11. Let $x(t)$ represent the amount of salt in the tank at time t. Note that $x(0) = 0.2 \times 50 = 1$.

$$\text{rate in: } 2 \times 0.1 \qquad \text{rate out:} 2 \times \frac{x}{50}$$

$$\frac{dx}{dt} = \frac{1}{5} - \frac{x}{25} \implies \frac{25}{5-x}\,dx = dt \implies -25\ln(5-x) = t + C$$

$$x(0) = 1 \implies C = -25\ln(4) \approx -34.66$$

The tank will reach 5% acidity when $x = 0.5 \times 50 = 2.5$.

$$-25\ln(5-2.5) = t - 34.66 \implies t \approx 11.75$$

Note that after 11.75 min, the amount of solution in the storage container will be $11.75 \times 2 = 23.5$. The total amount of acid in the system is $1 + 11.75 \times 0.2 = 3.35$ (counting the amount initially in the tank and the amount poured in). Of that $0.05 \times 50 = 2.5$ is now in the tank. That means there is $3.35 - 2.5 = 0.85$ acid in the storage container. The acidity of the solution in the storage container is:

$$\frac{0.85}{23.5} = 0.036 = 3.6\%$$