UNIQUE (PERIOD) PRIMES AND THE FACTORIZATION OF CYCLOMATIC POLYNOMIALS MINUS ONE

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ABSTRACT. Let \( b \) be an integer greater than one. For each prime \( p \) not dividing \( b \), the expansion of \( 1/p \) (radix \( b \)) has a period dividing \( p-1 \). Unique primes are those primes (not dividing \( b \)) for which no other prime has the same period. Let \( \Phi_n(X) \) be the \( n \)th cyclotomic polynomial. We develop criteria for when \( \Phi_n(X) - 1 \) is divisible by \( \Phi_k(X) \), and use this to add 18 new primes (and 10 new probable primes) to the list of the 29 previously known unique primes (base \( 10 \)). The divisibility properties found are useful in the search for primes of many forms.

1. DEFINITION OF UNIQUE PRIMES

Let \( p \) be a prime not dividing \( 10 \). The decimal expansion of the prime reciprocal \( 1/p \) has a period dividing \( p-1 \) (because the period of the prime \( p \) is also the order of the element 10 in the multiplicative group \( \mathbb{Z}_p \)). For example, the primes 3, 11, 37 and 101 are the only primes with periods 1, 2, 3 and 4 respectively. The pairs of primes 41 and 271, 7 and 13, and 239 and 4649 have periods 5, 6 and 7 respectively. In [24] Yates defined the unique primes to be those primes (such as 3, 11, 37 and 101) for which no other prime has the same period. This type of prime is very rare—there are only 18 of them less than \( 10^{50} \) (they are explicitly listed in [5]).

Given any positive integer \( n \), all the primes with period \( n \) must divide \( \Phi_n(10) \) where \( \Phi_n(x) \) is the \( n \)th cyclotomic polynomial:

\[
\Phi_n(X) = \prod_{1 \leq k \leq n-1, \gcd(n,k)=1} (X - \zeta_n^k) = \prod_{d|n} (X^d - 1)^{\mu(n/d)}, \quad \text{and} \quad X^n - 1 = \prod_{d|n} \Phi_d(X).
\]

(Here \( \zeta_n \) is any primitive \( n \)th root of unity and \( \mu \) is the Möbius function.) In general, not every factor of \( \Phi_n(b) \) is primitive (that is, does not divide \( \Phi_m(b) \) for any \( m < n \)), but those factors that are not primitive are easy to spot: they are the “intrinsic” factors and must divide the number \( n \) [4]. In conclusion, \( p \) is a unique prime with period \( n \) if and only if \( \frac{\Phi_n(10)}{\gcd(\Phi_n(10),n)} = p^k \) for some positive integer \( k \).

Notes: (1) To find the unique primes in other bases \( b \) we need only replace 10 with \( b \) in the discussion above. (2) It seems reasonable to conjecture that the quotient above is a true power (a power greater than one) only finitely often (and for \( b = 10 \), perhaps only when \( n = 1 \)). (3) When the period \( n \) itself is a prime, then \( \Phi_n(b) \) is the \( n \)th base \( b \) “repunit” \((b^n-1)/(b-1)\) (see [16,25]). Anytime a repunit is prime, it is a unique prime. These numbers have been screened for \( b \leq 12 \) with \( n \leq 1000 \) by Williams and Seah [18]; for \( b = 10 \) with \( n < 2000 \) by Williams and Seah.
[18,19], then for \( n < 16500 \) by Dubner [9,19]; and for \( b \leq 80 \) (with varying limits on \( n \)) by Dubner [9]; and of course for \( b = 2 \) (the Mersenne primes) for \( n < 797850 \) (and miscellaneous larger exponents), see the status page [23] of the *Great Internet Mersenne Prime Search* [22].

(4) Many other classical prime forms are subsumed by \( \Phi_n(b) \), for example the Fermat primes are \( \Phi_{2^{n+1}}(2) \) and the generalized Fermat primes \( \Phi_{2^{n+1}}(b) \)[2,11,21].

2. RESULTS OF SEARCH

Using the Fermat type primality tests, it is very easy to find unique probable primes. Using a special processor designed for large integer arithmetic [6,10,15] we screened the numbers \( \Phi_n(10) \) (*sans* intrinsic factors) up to \( n = 10000 \) (Harvey Dubner [8] did most of the screening for \( n < 6000 \)) and found them to be probable prime for the following values of \( n \):

\[
\begin{align*}
  n &= 385, 945, 1172, 1282, 1404, 1427, 1452, 1521, 1752, 1812, 1836, 2134, 2232, \\
  & \quad 2264, 2667, 3750, 3903, 3927, 4274, 4353, 6437, 6522, 6682, 8696, \text{ and } 9550.
\end{align*}
\]

as well as for the 29 (mostly smaller) values of \( n \) previously proven prime by others [19,24,25,26] (these may be found in table one). At the author’s request Morain [13] has shown that three of these numbers are prime (\( n = 1426, 1862 \) and 2134). Though Morain and has shown both a 1505 digit and a 1226 digit number prime using Atkin’s test [1,12], the size of most of these probable primes makes most of them well beyond the range the general primality proving routines. This leaves the classical tests based on the factorizations of \( N \pm 1 \)... (see [4,20]). Fortunately, we are able to show (using the theorems below) that \( \Phi_n(b)−1 \) is often divisible by \( \Phi_m(b) \) with \( m < n \), so we could use the extensive factorization tables which exist for the numbers \( b^n \pm 1 \) (such as [3,4,7]), and then complete the primality proofs for many of the probable primes above (see table one).

When factoring numbers defined by polynomials, we usually start by factoring the polynomials. For example, if \( n \) is a prime power (\( n = pr \)), or if \( n \) is a power of two times an odd prime power \( n = 2^sp^r \), then we have respectively

\[
\Phi_n(X) - 1 = X^{p^{r-1}} \frac{X^{p^{r-1}(p-1)} - 1}{X^{p^{r-1}} - 1} = X^{n/p} \prod_{d | \phi(n)} \Phi_d(X) \quad (n = p^r)
\]

\[
\Phi_n(X) - 1 = X^{n/2p} \prod_{d | \phi(n)} \Phi_d(X) \quad (n = 2^sp^r)
\]

In these cases \( \Phi_n(X) - 1 \) factors completely using cyclotomic polynomials. For other cases the factorizations are not complete. The first four such cases are the following.

\[
\Phi_{15}(X) - 1 = X \Phi_1(X) \Phi_2(X) \Phi_4(X) (X^3 - X^2 + 1)
\]
\[
\Phi_{21}(X) - 1 = X \Phi_1(X) \Phi_2(X) \Phi_6(X) (X^7 + X + 1)
\]
\[
\Phi_{30}(X) - 1 = X \Phi_1(X) \Phi_2(X) \Phi_4(X) (X^3 + X^2 - 1)
\]
\[
\Phi_{33}(X) - 1 = X \Phi_1(X) \Phi_2(X) \Phi_5(X) \Phi_{10}(X) (X^9 - X^8 + X^6 - X^5 + X^3 - X^2 + 1)
\]

The fact that \(\Phi_n(X) - 1\) is often divisible by \(\Phi_m(X)\) is shown by the following:

**Theorem.** Let \(n > 1\) be an integer and write \(n = \prod p_i^\alpha_i\) (where the \(p_i\) are distinct primes). Set \(R = \prod p_i\) and \(n = LR\). Let \(p\) be any prime divisor of \(n\). Then \(\Phi_n(X) - 1\) is divisible by \(\Phi_k(X)\) for the following values of \(k\):

1. \(k \mid L\) whenever \(R\) is not a prime,
2. \(k \mid 2L, k \not\mid L\) whenever \(R\) is not 2 or twice a prime,
3. \(k \mid (p-1)L\) whenever \(pk \neq (k.L)R\).
4. \(k \mid ((p+1), \phi(R/p))L\) whenever \(pk \neq (k.L)R\).

(Here \(\phi\) is Euler’s totient function.) We will present the proof in the next section.

**Examples:**
1. When \(N = \Phi_{2264}(10)\), this theorem tells us that \(\Phi_m(10)\) divides \(N-1\) for every divisor \(m\) of 1128 except 8. The product of these terms is exactly \(N-1\) and the known factors of these terms are more than sufficient to complete the primality proof. This 1128 digit number is currently the largest known unique prime (base 10). 2. When \(N = \Phi_{3750}(10)\), the theorem above tells us that \(\Phi_m(10)\) divides \(N-1\) for each of the divisors \(m\) of 500. The product of these terms is slightly over 60% of \(N\), and enough of their factors are known to complete the primality proof. This number has the longest period of any known (base 10) repunit-prime. 3. When \(N = \Phi_{1521}(10)\), the theorem above tells us that \(\Phi_m(10)\) divides \(N-1\) for each of the divisors \(m\) of 468 except 9 and 117. This would be enough factors (the product of the values of these cyclotomic polynomials is greater than the cube root of \(N-1\)) but currently not enough factors of \(\Phi_{468}(10)\) are known. However it is easy to find enough small factors of the cofactor to complete the primality proof.

Again, using this theorem we were able to complete the primality proofs for all but ten of the probable primes in the list above. These probable primes are marked with brackets in table 1.

### 3. Necessary and Sufficient Conditions that \(\Phi_m(b)\) Divides \(\Phi_n(b)-1\)

Rather than focus on whether \(\Phi_k(X)\) divides \(\Phi_n(X) - 1\), we address the equivalent question of when \(\Phi_n(\zeta_k) = 1\) for a primitive \(k\)th root of unity \(\zeta_k\). This first theorem tells us that \(k\) must divide \(\phi(n)\).

**Theorem 1.** Let \(n > 1\). \(\Phi_n(\zeta_k)\) is a real number if and only if \(k = n\) or \(k \mid \phi(n)\).

**Proof.** Choose \(\alpha\) so that \(\zeta_k = e^{2\alpha i}\) and then set \(\zeta_{2k} = e^\alpha i\). A quick calculation shows...
First suppose that \( k \nmid n \). Using the definition of \( \Phi_n \) we see that

\[
\Phi_n(\zeta_k) = \prod_{d|n} (\zeta_k^d - 1)^{\mu(n/d)} = \prod_{d|n} (2i \zeta_{2k}^d \sin d\alpha)^{\mu(n/d)}
\]

\[
= (2i)^{\sum \mu(n/d)} \sum d|n \zeta_{2k}^d \prod_{d|n} (\sin d\alpha)^{\mu(n/d)}
\]

\[
\Phi_n(\zeta_k) = \zeta_{2k}^{\phi(n)} \prod_{d|n} (\sin d\alpha)^{\mu(n/d)}
\]

(here the sums, like the products, are taken over the divisors \( d \) of \( n \)). This shows that \( \Phi_n(\zeta_k) = \zeta_{2k}^{\phi(n)} F_n(\alpha) \) where \( F_n(\alpha) \) is a real valued function. Thus \( \Phi_n(\zeta_k) \) is real if and only if \( \zeta_{2k}^{\phi(n)} \) is real, which happens if and only if \( k | \phi(n) \).

Now suppose \( k | n \),

\[
\Phi_n(X) = \Phi_{n/k}(X^k) \prod_{d|n \atop k \nmid d} (X^d - 1)^{\mu(n/d)}
\]

so arguing as above,

\[
\Phi_n(\zeta_k) = \Phi_{n/k}(1) \zeta_{2k}^{\phi(n)/k} \prod_{d|n \atop k \nmid d} (\sin d\alpha)^{\mu(n/d)}.
\]

Here \( \Phi_n(\zeta_k) \) is real iff either \( \Phi_{n/k}(1) = 0 \) (that is, \( n = k \)) or \( k \mid (\phi(n) - k\phi(n/k)) \), proving the theorem.

Now that we know \( k \mid L\phi(R) \) is necessary, we give several sufficient conditions, starting with the theorem quoted in the previous section (properly rephrased):

**Theorem 2.** Let \( n > 1 \) be an integer and write \( n = \prod p_i^{\alpha_i} \) (where the \( p_i \) are distinct primes). Set \( R = \prod p_i \) and \( n = LR \). Let \( p \) be any prime divisor of \( n \). \( \Phi_n(\zeta_k) = 1 \) for all \( k \) such that

(i) \( k \mid L \) whenever \( R \) is not a prime,
(ii) \( k \mid 2L, k \nmid L \quad \text{whenever} \quad R \text{ is not 2 or twice a prime}, \)

(iii) \( k \mid (p-1)L \quad \text{whenever} \quad pk \neq (k,L)R. \)

(iv) \( k \mid ((p+1,\phi(R/p))L \quad \text{whenever} \quad pk \neq (k,L)R. \)

**Proof.** For all \( X \) we have \( \Phi_n(X) = \Phi_R(X^L) \), so for any integer \( k \) and any primitive \( k \)th root of unity \( \zeta_k \) we have \( \Phi_n(\zeta_k) = \Phi_R(\zeta_k^L) \). If \( k \mid L \) and \( R \) is not a prime power, then \( \Phi_n(\zeta_k) = \Phi_R(1) = 1 \), giving (i). If \( k \mid 2L \) but \( k \nmid L \), then \( \Phi_n(\zeta_k) = \Phi_R(-1) \) which is 1 if \( R \) is not 2 or twice an odd prime. This, with (i), shows (ii).

Now suppose \( p \) is any prime dividing \( R \). When the denominator is not zero we have

\[
\Phi_n(X) = \frac{\Phi_{R/p}(X^{pL})}{\Phi_{R/p}(X^L)},
\]

so if \( k \) divides \( pL-L \), and \( pk \neq (k,L)R \), then

\[
\Phi_n(\zeta_k) = \frac{\Phi_{R/p}(\zeta_k^{pL})}{\Phi_{R/p}(\zeta_k^L)} = \frac{\Phi_{R/p}(\zeta_k^L)}{\Phi_{R/p}(\zeta_k^L)} = 1,
\]

showing (iii). Finally, if instead \( k \) divides \( pL+L \) (and again \( pk \neq (k,L)R \)), then because \( \Phi_{R/p} \) is a reciprocal polynomial we have

\[
\Phi_n(\zeta_k) = \frac{\Phi_{R/p}(\zeta_k^{-L})}{\Phi_{R/p}(\zeta_k^L)} = \zeta_k^{-L}\phi(R/p).
\]

This quotient will be one if \( k \mid L\phi(R/p) \), which is (iv).

It is interesting to note that the simple criteria of the above theorem accounts for almost all of the small solutions to \( \Phi_n(\zeta_k) = 1 \). For example, of the 21206 solutions to \( \Phi_n(\zeta_k) = 1 \) with \( n \leq 2000 \), this theorem omits only the following eighteen solutions:

\[
(n,k) = (385,24), (455,24), (627,24), (770,24), (910,24), (1105,24), (1155,24), (1254,24), (1295,24), (1365,24), (1419,24), (1540,48), (1705, 24), (1729,24), (1820,48), (1881,72), (1925,24), (1925,120).
\]

(These 21206 solutions were found using a *Mathematica* program.) All of the solutions listed above are among those found by the following results.

**Corollary 3.** Let \( p \equiv 19, q \equiv 11 \mod 24 \), and suppose \( 3pq \mid n \); then \( \Phi_n(\zeta_k) = 1 \) for all \( k \mid 24L \) except when \( k/(k,L) = 8 \).

(For example, \( \Phi_{3.11.19.m}(\zeta_{24}) = \Phi_{3.19.83.m}(\zeta_{24}) = 1 \) for all positive integers \( m \) not divisible by 3.)
Proof. Theorem 2 shows $\Phi_{3.11.19}(\zeta_k) = 1$ for $k = 1, 2, 3, 4, 6$ and 12; and it is easy to check that $\Phi_{3.11.19}(\zeta_k) = 1$ for $k = 24$ but not for $k = 8$, so it follows that $\Phi_{3pq}(\zeta_k) = 1$ for $k \mid 24, k \neq 8$.

Next, for all primes $r \mid R$, we have $\Phi_R(\zeta_k) = \Phi_{R/r}(\zeta_{r^k})/\Phi_{R/r}(\zeta_k)$, so it follows that $\Phi_R(\zeta_k) = 1$ whenever $3 \mid R$ (with the same condition on $k$). Finally,$$
\Phi_n(\zeta_k) = \Phi_R(\zeta_k) = \Phi_R(\zeta_{k/(k,L)}) = 1$$whenever $k/(k,L)$ divides 24, but is not 8.

Theorem 4. Let $p$, $q$, and $r$ be distinct primes dividing $n$. Write $n = RL$ as above. If

$k \mid (p^2-1, q^2-1, pqr-1) L$ or $k \mid (p^2-1, q^2-1, pqr+1, L\phi(R/r)) L$,

then $\Phi_n(\zeta_k) = 1$.

(For example, $\Phi_{5.7.11.13.17.19.23.29.31.37.41}(\zeta_{24}) = \Phi_{5.13.17.19.23.29.31.37.41}(\zeta_{24}) = \Phi_{5.11.13.17.19.23.29.31.37.41}(\zeta_{24}) = \Phi_{5.7.11.13.17.19.23.29.31.37.41}(\zeta_{24}) = 1$ for all positive integers $m$.)

Proof. Let $k$ be any divisor of $(p^2-1, q^2-1, pqr-1)$. By the choice of $k$, we have the following congruences

$$1 \cdot r \equiv pq, \ p \cdot r \equiv q, \ q \cdot r \equiv p, \ pq \cdot r \equiv 1 \pmod{k}.$$ 

That is, multiplication by $r$ just permutes the following sets modulo $k$.

$$A_{+}(pq) = \{ d \mid pq : \mu(d) = 1 \}$$

$$A_{-}(pq) = \{ d \mid pq : \mu(d) = -1 \}$$

Now let $s$ be any other prime divisor of $n$, $(s \nmid pqr)$. Again multiplication by $r$ will just permute the two sets $A_{\pm}(pqs) ...$ so this is true for the sets $A_{\pm}(R/r)$.$$
\prod_{d \in A_{+}(R/r)} \frac{(r^d \zeta_k - 1)}{(\zeta_k - 1)} = \prod_{d \in A_{+}(R/r)} \frac{(r^d \zeta_k - 1)}{(\zeta_k - 1)}$$

which shows $\Phi_{R/r}(\zeta_k) = \Phi_{R/r}(\zeta_{r^k})$ and $\Phi_{R}(\zeta_k) = \Phi_{R/r}(\zeta_{r^k})/\Phi_{R/r}(\zeta_k) = 1$. Finally,$$
\Phi_n(\zeta_k) = \Phi_R(\zeta_k) \text{ is one iff } \Phi_R(\zeta_{k/(L,k)}) \text{ is, which we have shown happens when } k/(L,k) \text{ divides } (p^2-1, q^2-1, pqr-1), \text{ completing the proof of the first case.}$
If instead we have that \( k \mid (p^2-1, q^2-1, pqr+1) \), then multiplication permutes and changes the sign of the elements in the lists \( A_\pm \), so we conclude (almost as above)

\[
\Phi_{R/r}(\zeta_k^r) = \Phi_{R/r}(\zeta_k^{r-1}) = \zeta_k^{-\phi(R/r)} \Phi_{R/r}(\zeta_k)
\]

(\( m \) square-free, relatively prime to \( pqr \)) and

\[
\Phi_n(\zeta_k) = \Phi_{R/r}(\zeta_k^r) / \Phi_{R/r}(\zeta_k) = \zeta_k^{-\phi(R/r)}
\]

completing the proof in the second case. \(\square\)

**Theorem 5.** Let \( p, q \) and \( r \) be distinct primes dividing \( n \), with \( p \) and \( q \) odd. Write \( n = RL \) as above. If

\[
k \mid (q \pm 1, r(p \pm 1), \phi(rpq)/2) \text{ and } k \nmid rpq
\]

then \( \Phi_n(\zeta_k) = 1. \)

(For example, \( \Phi_{7\cdot23\cdot71\cdot m}(\zeta_{24}) = \Phi_{19\cdot47\cdot73\cdot m}(\zeta_{24}) = 1 \) for all positive integer \( m \).)

**Proof.** If \( 2k \mid (q \pm 1) \), then \((1, pq) \equiv (\mp q, \mp p) \pmod{2k}\). If instead \( k \mid (q \pm 1) \) but \( 2k \nmid (q \pm 1) \), \( p \neq 2 \), then \((1, pq) \equiv (k \mp q, k \mp p) \pmod{2k}\). Either way,

\[
\sin \frac{\pi}{k} \sin \frac{pq\pi}{k} = \sin \frac{p\pi}{k} \sin \frac{q\pi}{k}.
\]

Similarly, \( k \mid r(p \pm 1) \), \( q \neq 2 \), implies

\[
\sin \frac{r\pi}{k} \sin \frac{rq\pi}{k} = \sin \frac{r\pi}{k} \sin \frac{rpq\pi}{k}.
\]

Together these show \( \Phi_{rpq}(\zeta_k) = \zeta_{2k}^{\phi(rpq)} \prod_{d \mid rpq} (\sin d\pi/k)^{\mu(rpq/d)} = 1. \) \(\square\)

(Nota the problem of finding all rational solutions to \( \sin \pi x_1 \sin \pi x_2 = \sin \pi x_3 \sin \pi x_4 \) has been completely solve by Myerson [14].)

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<td>441</td>
<td>56*</td>
<td>[8696, 4344]</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>120</td>
<td>33</td>
<td>38*</td>
<td>1521</td>
<td>936</td>
<td>57*</td>
<td>[9550, 3801]</td>
<td></td>
</tr>
</tbody>
</table>

Notes: The unique primes (except for \(n=1, p=3\)) are given by \(\frac{\Phi_n(10)}{\gcd(\Phi_n(10),n)}\).

The primes whose periods are indicated by \# were proven prime by Morain [13].
The primes whose periods are indicated by * were proven by the by author.
The numbers in square brackets [ ] correspond to probable primes.
All periods \(n \leq 10000\) were tested.