You are allowed one 3 × 5 index card. Show all work.

1 (15 points). Give the annihilator of each of the following expressions:

\[ 4x^2 + x^2 \cos(3x) \]

\[ 3x^3 e^{5x} \sin(4x) \]

\[ x + e^{10x} \]

2 (20 points). Find the general solution to the given differential equations.

\[ y'' - 8y' + 16y = 0 \]

\[ y''' - y = 0 \]

\[ y''' - y'' - 8y' + 12y = 0 \]

\[ (D^2 + 9)^3(D - 2)^2(D^2 + 4)[y] = 0 \]

3 (10 points). Find the general solution to the differential equation:

\[ y'' - 2y' + y = 8e^{3x} + 4 \sin x \]

4. (5 points) What are the critical points of the system of differential equations:

\[
\begin{align*}
x' &= x^2 - x \\
y' &= 2x - y
\end{align*}
\]

5. (20 points) (i) Use the method of undetermined coefficients to set up a particular solution the the differential equation. You do not need to solve for the coefficients themselves. (ii) Use
the method of annihilators to convert the differential equation to a higher order homogeneous differential equation. You do not need to solve this differential equation.

\[ y'' - y = 5e^x \]

\[ y'' + 7y = 2\sin x + e^{3x} \]

6. **(10 points)** For the given system of differential equations, find and solve the “phase-plane equation” to find the (nonparametrized) solution curves:

\[
\begin{align*}
  x' &= x^2 + 1 \\
  y' &= xy^2
\end{align*}
\]

7. **(10 points)** An object with a mass of 2 kg is dropped from a great height. The magnitude of the force on the object due to air resistance is \(|v|/20\). What is the terminal velocity of the object?

8. **(10 points)** Consider boxes coupled by springs in the arrangement depicted below. Here’s the catch: at time \( t = 0 \) it starts raining, and the boxes begin to fill up water, so each of their masses increase at a rate of .001 kg/s. Set up a system of equations to model this system. You do not need to solve the system.
Solutions

1. 

(a) \[ L = D^3(D^2 + 9)^3 \]

(b) \((x - 5 + 4i)(x - 5 - 4i) = x^2 - 10x + 41 \implies L = (D^2 + 10D + 41)^4 \]

(c) \[ L = D^2(D - 10) \]

2. 

(a) \[ r^2 - 8r + 16 = 0 \implies (r - 4)^2 = 0 \implies r = 4 \text{ (mult 2)} \]

\[ \implies y = C_1e^{4x} + C_2xe^{4x} \]

(b) \[ r^3 - 1 = 0 \implies (r - 1)(r^2 + r + 1) = 0 \implies r = 1, \frac{-1 \pm i\sqrt{3}}{2} \]

\[ \implies y = C_1e^x + C_2e^{-x/2}\sin(\sqrt{3}x/2) + C_3e^{-x/2}\cos(\sqrt{3}x/2) \]

(c) \[ r^2 - r^2 - 8r + 12 = 90 \implies (r - 2)^2(r + 3) = 0 \]

\[ \implies y = C_1e^{2x} + C_2xe^{2x} + C_3e^{3x} \]

(d) \[ y = C_1\sin(3x) + C_2\cos(3x) + C_3x\sin(3x) + C_4x\cos(3x) + C_5x^2\sin(3x) + C_6x^2\cos(3x) + C_7e^{2x} + C_8xe^{2x} + C_9\sin(2x) + C_{10}\cos(2x) \]

3. The general solution to the homogeneous equation is \( y = C_1e^x + C_2xe^x \). Now to find a particular solution to the nonhomogeneous equation of the form:

\[ y = Ae^{3x} + B\sin(x) + C\cos(x) \]

Computing derivatives and plugging into the equation, we find that \( A = 2, B = 0 \) and \( C = -2 \). Therefore, the general solution is:

\[ y = C_1e^x + C_2xe^x + 2e^{3x} - 2\cos(x) \]

4. 

\[ x^2 - x = 0 \implies x = 1, 0 \]

\[ 2x - y = 0 \implies y = 2x \]

Therefore, the critical point are \((0, 0)\) and \((1, 2)\).
5.

(a1) \( y = Axe^x \)

(a2) \( (D^2 - 1)[y] = 5e^{5x} \Rightarrow (D - 1)(D^2 - 1)[y] = 0 \Rightarrow (D - 1)^2(D + 1)[y] = 0 \)

(b1) \( y = A \sin x + B \cos x + Ce^{2x} \)

(b2) \( (D^2 + 7)[y] = 2 \sin x + e^{3x} \Rightarrow (D^2 + 1)(D^2 + 7)[y] = 0 \)

6.

\[
\frac{dy}{dx} = \frac{y'}{x'} = \frac{xy^2}{x^2 + 1} \Rightarrow \int \frac{dy}{y^2} = \int \frac{x}{x^2 + 1} \, dx
\]

sol: \(-y^{-1} = \frac{1}{2} \ln(x^2 + 1) + C\)

7.

\[
v' = -2 \times 9.81 + \frac{v}{20} \Rightarrow v = 392.4 + Ce^{-t/40}
\]

\(v(0) = 0 \Rightarrow C = 392.4 \Rightarrow v(t) = 392.4(1 - e^{-t/40})\)

The terminal velocity is 392.4.

8.

\[
\begin{cases}
(1 + 0.001t)x'' = -x + 2(y - x) \\
(2 + 0.001t)y'' = -(y - x)
\end{cases}
\]