A Common Error: $\sqrt{a^2+b^2}$ versus $a+b$

Standards Addressed

1. A2.A.REI.A.2, M.A2.A.REI.A.2: Solve rational and radical equations in one variable, and identify extraneous solutions when they exist.
2. The Standards for Mathematical Practice, especially: 1. Make sense of problems and persevere in solving them; 2. Reason abstractly and quantitatively; and 3. Construct viable arguments and critique the reasoning of others.

Materials Needed

Enough copies of the work sheet for you students (or write the two questions on the board). Math journals (if you use them).

Before conducting the activity

Make sure that you understand the mathematics and graphs.

Conducting the Activity

1. Ask the first question and follow the suggestions on the teacher's guide (third page).
2. Ask the second question and follow the suggestions on the teacher's guide (third page).
3. Do not skip having the students write! Writing is important for several reasons: (a) it requires that the think about and organize what they are doing—which is always useful, (b) it is part of the state "Literacy Skills for Mathematical Proficiency" and (c) it is key to standard (3) of the "Standards for Mathematical Practice." If you are working on that today, then be sure to have your students read and critique each others work.

Variations

Use $\sqrt[3]{x^3+1}$ and $x+1$ (so the behavior is different for negative numbers x.)

Extension

Ask students for which integers n does

$$(1+2+\cdots+n)^2 = 1^2+2^2+\cdots+n^2?$$

(Just $n=1$). For which integers n does

$$(1+2+\cdots+n)^2 = 1^3+2^3+\cdots+n^3?$$

(All $n>0$.)

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Student Worksheet

1. Determine when $\sqrt{x^2+1^2} = x+1$ and show your answer is correct in at least two ways. Write your work down here or in your journal—whichever you teacher prefers. Show each step.

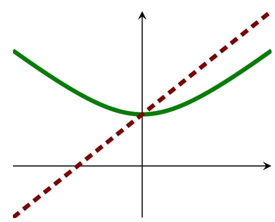
2. Let b be a real number. When is $\sqrt{x^2+b^2} = x+b$? Write you argument down showing each step clearly. Can you show that your answer is correct in at least two ways? If so, do so. If not, write clearly why not. (Write your work down here or in a journal.)

Teacher Guide

1. Determine when $\sqrt{x^2+1^2} = x+1$ and show your answer is correct in at least two ways. Write your work down here or in your journal—whichever you teacher prefers. Show each step.

Let your students discuss this awhile, then ask questions to lead them to at least an algebraic solution and a graphical solution. An algebraic solution could be as follows.

$$\begin{aligned}
\text{The original problem:}&\quad \sqrt{x^2+1} = x+1. && (1)\\
\text{Square both sides:}&\quad x^2+1 = (x+1)^2. && (2)\\
\text{Expand the right side:}&\quad x^2+b1 = x^2+2x+1. && (3)\\
\text{Subtract } x^2+b^2 \text{ from both sides:}&\quad 0 = 2x. && (4)\\
\text{Divide both sides by 2:}&\quad x = 0. && (5)
\end{aligned}$$



Ask the students to compare the relative merits of these two methods. E.g., can all equations be solved algebraically? (Try $3^x = x^3$, $x^x = 3$ or $x \log x = 3$.) Can you always tell the solution from the graph? (Wrong window, zeros too close together, ...) When graphing, how can you know you found **all** of the solutions?

Try to get them to think about when each method is must appropriate. (This does not have a succinct answer–but that is often te case in the real world.)

2. Let y be a real number. When is $\sqrt{x^2+y^2} = x+y$? Write you argument down showing each step clearly. Can you show that your answer is correct in at least two ways? If so, do so. If not, write clearly why not. (Write your work down here or in a journal.)

A solution might start as follows.

$$\begin{aligned}
\text{The original problem:}&\quad \sqrt{x^2+y^2} = x+y. && (1)\\
\text{Square both sides:}&\quad x^2+y^2 = (x+y)^2. && (2)\\
\text{Expand the right side:}&\quad x^2+y^2 = x^2+2xy+y^2. && (3)\\
\text{Subtract } x^2+y^2 \text{ from both sides:}&\quad 0 = 2xy. && (4)
\end{aligned}$$

If the have not made it to here–work with the class to reach this step.

Explore how the students interpret $2xy = 0$. For example, a reasonable error is to interpret this as meaning "either $x = 0$ or $y = 0$." Prompt their understanding (and have them prompt each other) with questions like "is $(x, y) = (-1, 0)$ a solution?" Notice I purposely omitted "a solution to our original problem." Instead you might ask them "you are seeking a solution to what?"

A correct answer is:

$$\begin{cases} y = 0 \text{ and } x \geq 0, \text{ or} \\ y \geq 0 \text{ and } x = 0. \end{cases}$$

This can be written many ways, including changing one of the $\geq$ to $>$. Have them check each others' answers.

Some of the solutions to $2xy = 0$ were extraneous (not solutions to $\sqrt{x^2+y^2} = x+y$). Have the students find (and write down) exactly which step in their solution generated these false solutions (such as $(x, y) = (-1, 0)$). [The extraneous solutions are introduced when we square (step (2)).]

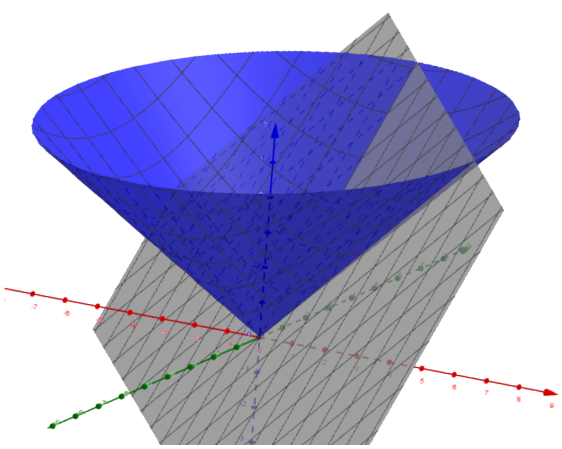
Solving this graphically is difficult because it requires a 3d graph. (Can your students tell you why it is 3d even though we only have the two variables x and y?) As a teacher you should use GeoGebra to graph this. To do so, select the "3D Graphics" perspective and hide the xy-plane using the appropriate icon.

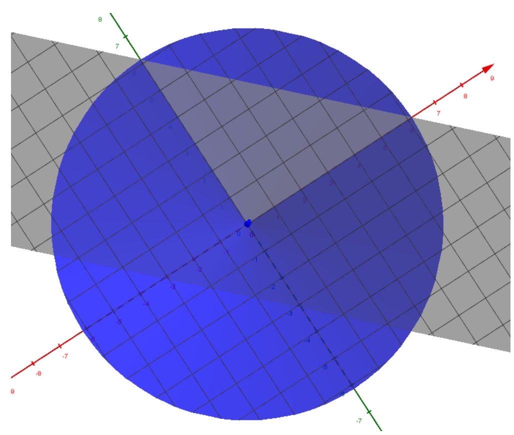
In the input bar type `sqrt(x^2+y^2)` to graph $z = \sqrt{x^2+y^2}$.

Ask the student what the shape is. Consider having them record their answer to force them to commit. (A right circular cone, standard 8.G.C.7, ...).

Ask why that equation defines a cone. Use questions as hints, for example, for a fixed value of z what shape do you get? If $x = 0$, what shape to you get? These are literally conic sections standards P.A.C.*).

Now we are going to graph $z = x + y$, but first ask students what shape this is going to be. Then enter `x+y` in the input bar. Change the color of one of your two graphs so the intersection os easier to see.





What is the shape of the intersection? Look at it straight down the z-axis and see that it is the two rays we found in the algebraic answer:

$$\begin{cases} y = 0 \text{ and } x \geq 0, & \text{or} \\ y \geq 0 \text{ and } x = 0. \end{cases}$$