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PRIMES IN PI

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1. Introduction

> "If I let my fingers wander idly over the keys of a typewriter it might happen that my screed made an intelligible sentence. If an army of monkeys were strumming on typewriters they might write all the books in the British Museum."
>
> Arthur S. Eddington, "The Nature of the Physical World" p.72 (1928)

It is generally believed that the digits of π are randomly distributed. It occurred to us that if monkeys might be so productive with random letters then the random digits of π might be productive in generating those most interesting of numbers, primes. Fortunately, it is considerably easier to deal with computers than an army of monkeys.

2. Monkeys

First we illustrate the process of generating primes by using the first 10 digits of $\pi = 3.141592653$, and ignoring the decimal point, scan the sequence of digits for single digit primes, then 2-digit primes, then 3-digit primes, etc., up to 10- digits in this case.
1-digit primes: 3, 5, 2, 5, 3
2-digit primes: 31, 41, 59, 53
3-digit primes: 653
4-digit primes: 4159
5-digit primes: 14159
6-digit primes: 314159
7-digit primes: 1592653
Total primes = 14

In what follows most of the calculations were made with UBASIC, A truly amazing PC public domain program oriented toward number theory. It can calculate π to 4000 digits in about 20 microseconds on a Pentium/200.

To determine if a number N is prime we first check if it has any small divisors up to 100,000. If not, we then check if the number satisfies the Fermat equation,

$$b^{N-1} \equiv 1 \pmod{N}, \quad \text{for } b = 3 \text{ and } 13.$$

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Table 1. Primes in π and e

	π	e
Number of digits	1000	1000
Number of 1-digit primes	398	391
Number of 2-digit primes	225	203
Number of 3-digit primes	136	141
Number of 4-digit primes	104	124
Number of 5-digit primes	87	90
Total primes 2-digit or greater	2345	2284
Smallest missing primes	103 107 131 149 157	109 131 137 139 179
Largest prime (starting digit)	932-digits (46)	963-digit (30)

Note: Primes less than 10 digits are truly prime.
Primes greater than 10 digits are probable primes (PRP).

If it satisfies this equation, it is almost certain that N is prime. For many large numbers it simply is not practical to prove absolute primality. As an example, the Fermat test for a 300-digit number takes less than a second, while confirming absolute primality takes over an hour. However, we have shown that all primes 10 digits or less are truly prime. For the rest of this article we will say a number is prime if it meets the above criteria.

3. Results

π is certainly the most important constant in science and mathematics, but $e = 2.71828...$, the base of natural logarithms, is a close second. We therefore performed the same calculations on e as we did on π. The results of searching for primes in the first 1000 digits of each is shown in Table 1.

The first reaction of most people is surprise at how many primes are included in π and e, and at the size of the largest primes. This is entirely to be expected theoretically as we will show in the next section. However, to obtain a feeling as to how a string of random numbers behaves, we statistically investigated the generation of primes by randomly produced digits. For practical reasons we limited the number of digits to 100 for each test and ran 100 tests. The results are shown in Table 2. Note that the number of primes for 100 digits of π and e fall well within the range that would be expected for random digits.

4. Theory

Out of 90 possible 2-digit numbers, 21 of them are prime, so that the probability of a random 2-digit number being prime is 21/90. By actual count from prime tables the exact probabilities for 3-digit primes, 4-digit primes, etc. can be calculated. For large numbers of digits, tables are not available but the probability can be calculated using the Prime Number theorem which gives the approximate density of primes. According to the Prime Number Theorem, the probability that a number N is prime is $1/\log(N)$. This information is collected in Table 3. Note how close the approximate probability (column 4) is to the exact probability for $k > 1$. From here on we will ignore single digit primes.

The approximate probability assumes that the prime probability is based on the prime density in the center of the range of interest. The expected number of k-digit

Table 2. Number of Primes in Random Digits

(100 Tests- Not Including Single Digits)

Range	Freq	Cum
70-79	1	1
80-89	1	2
90-99	0	2
100-109	8	10
110-119	14	24
120-129	15	39
130-139	20	59
140-149	18	77
150-159	12	89
160-169	6	95
170- 179	4	99
180-179	1	100

Note: Mean = 134.88;
Standard Deviation = 20.58
Primes in first 100 digits of π = 145;
Primes in first 100 digits of e = 131

Table 3. k-Digit Prime Probabilities

	k-Digit Primes	Exact Probability	Approximate Probability $1/\log(10^k/2)$
1	4	.4	.62
2	21	.2333	.2556
3	143	.1589	.1609
4	1061	.1179	.1174
5	8364	.0929	.0924
6	68906	.0766	.0762

primes is the product of the number of possible primes by the probability of each being prime.

Expected number of k-digit primes =

$$\frac{n-k+1}{\log(10^k/2)} = \frac{n-k+1}{k\log(10)-\log(2)}.$$

Denoting $E(n,a,b)$ as the expected number of primes with at least a digits and up to b digits, out of a sequence of n random digits,

$$E(n,a,b) = \sum_{k=a}^{b} \frac{n-k+1}{k\log(10)-\log(2)}. \tag{4.1}$$

Next we will derive approximate expessions valid for large n.

Replacing the summation sign with an integral, and with some simplification, the above equation becomes

Table 4. Expected Number of Primes

Number of Digits n	Minimum Prime Digits a	Maximum Prime Digits b	Expected Primes $E(n,a,b)$	Actual Primes in π
100	2	100	149	145
1000	2	2	255	225
1000	2	1000	2478	2345
1000	900	1000	2.4	1
1000	800	1000	10.2	14
10000	2	10000	34756	
10000	9000	10000	23.4	
100000	2	100000	447553	
100000	90000	1000000	233	
10^9	2	10^9	$9 \cdot 10^9$	
$206 \cdot 10^9$	$200 \cdot 10^9$	$206 \cdot 10^9$	38700000	

$$E(n,a,b) = \frac{(n+1)[\log(b) - \log(a)] - (b-a)}{\log(10)}. \tag{4.2}$$

Letting $b = n$ and $a = 2$, so that single digit primes are omitted,

$$E(n) = E(n,2,n) = \frac{(n+1)\ [\log(n) - \log 2] - n + 2}{\log(10)}. \tag{4.3}$$

For large n, this approaches

$$E(n) = \frac{(n+1)\ \log(n)}{\log(10)} \approx n \log_{10}(n). \tag{4.4}$$

a very convenient formula.

By using the appropriate equations, we derived Table 4. We tried to use Eq 4.1 whenever possible since it is most exact.

At the time this article was written the record for the number of digits produced for the decimal expansion of was 206 billion. From the last entry in Table 4. we can expect that there are over 38 million primes in the known expansion of π that have at least 200 billion digits. Finding any of them is far, far, beyond existing technology and theory but it is interesting to know they are there.

Next, consider a particular 9-digit prime such as 100000007. The chance of any nine consecutive digits matching this prime is 10^9. However there are approximately $2 \cdot 10^{11}$ possible 9-digit sequences that could equal the prime. Thus in the 200 billion digit expansion we can expect this prime to appear an average of 200 times. Without showing the detailed calculations, the chance of it not appearing at all is vanishingly small. Even though there are about 10^8 different primes with nine digits it is a virtual certainty that all the 9-digit primes will appear at least once in the known digits of π. The same type of reasoning will show that we can expect that many 10-digit primes will be missing.

Table 5. A Short Chronology of the Calculation of π

Person	Date	Decimal Places	Comments
Rhind Papyrus	c. 1650	1	
Archimedes	c. 250 BC	3	in/circumscribed 96-gons
Liu Hui	263	5	inscribed 3072-gon
Tsu Chung-Chi	c. 470	6	refined Hui's method
Ludolph van Ceulen	1580	14	refined Archimedes method
Abraham Sharp	1700	71	Gregory's series
John Machin	1706	100	Machin's formula
Fautet de Lagny	1719	113	
Georg von Vega	1794	136	
William Rutherford	1824	152	
Johann Dase	1844	205	
Thomas Clausen	1847	248	
William Rutherford	1853	440	
Richter	1855	500	
William Shanks	1873	527	
D. F. Ferguson	1945	560	
D. F. Ferguson	1946	620	
D. F. Ferguson	1947	808	Desk calculator
Smith and Wrench	1947	818	Desk calculator
Smith and Wrench	1949	1,120	Desk calculator
George Reitwiesner	1949	2,037	ENIAC 70 hours
Nickolson and Jeenel	1955	3,089	NORC computer
G. E. Felton	1957	7,480	Ferranti computer
Genuys	1958	10,000	IBM 704 100 minutes
(Genuys' program)	1959	16,167	IBM 704
D. Shanks and Wrench	1961	100,000	IBM 7090 9 hours
D. Shanks and Wrench	1966	250,000	IBM 7030
Guilloud and Filliatre	1967	500,000	CDC 6600
Guilloud and Bouyer	1973	1,000,000	CDC 7600 24 hours
Kanada et. al.	1983	16,000,000	HITAC M-280H, AGM 30 hours
W. Gosper	1985	17,000,000	Symbolics 3670,cont,fract.
Bailey	Jan. 1986	29,360,000	Cray 2 28 hours
Y. Kanada	Jul .1987	33,554,432	s-810/20 6 hours
Y. Kanada	1988	201,326,000	s-810/20 1.5 days
D. & G. Chudnovsky	Jan. 1989	480,000,000	Cray & IBM machines
D. & G. Chudnovsky	Aug. 1989	1,011,196,691	Special purpose machine
Y. Kanada	Nov. 1989	1,073,740,000	
Y. Kanada & Tamura	Nov. 1989	1,073,741,799	
D. & G. Chudnovsky	Aug. 1991	2,160,000,000	Special purpose machine
D. & G. Chudnovsky	May 1994	4,044,000,000	
Takahasi & Kanada	Jun. 1995	3,221,225,466	
Takahasi & Kanada	Aug. 1995	4,294,967,266	
Takahasi & Kanada	Oct. 1995	6,442,450,938	
Takahasi & Kanada	1997	51,539,600,000	
Takahasi & Kanada	Sept. 1999	206,158,430,000	Hitachi SR8000

5. How is the Decimal Expansion of Calculated?

In the earliest days, a value of 3 or $\sqrt{10}$ was used for π. As mathematical knowledge exploded under the Greeks, much better estimates of π were derived using perimeters of circles calculated using inscribed and circumscribed n-gons[1]. This period is shown as the second division in Table 5.

The n-gon method was used up until the eighteenth century when Gregory found his expansion for the inverse tangent:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots .$$

Using Gregory's series, Abraham Sharp began the third period shown in Table 5 by writing

$$\frac{\pi}{6} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{3^2\cdot 5} - \frac{1}{3^3\cdot 7} + \cdots\right).$$

and then using this series to calculate π to 71 decimal places. Next John Machin used Gregory's series in the form that bears his name

$$\pi = 16\tan^{-1}\frac{1}{5} - 4\tan^{-1}\frac{1}{239}$$

to calculate π to 100 decimal places in 1706.[1] This type of inverse tangent formula was used by all the digit collectors for the next 267 years (see the histories in [1-4]).[2]

All but one of the most recent calculations[3] have used algorithms based on the arithmetic-geometric mean or (AGM). The AGM of two numbers a, b is defined as the mutual limit of the following two sequences:

$$a \geq= b > 0, a_0 = a, b_0 = b, \quad \text{and} \quad a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n}\ .$$

These sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit, $M(a_0, b_0)$, determined uniquely by a_0 and b_0. It can be shown that, for $a_0 = 1$ and $b_0 = +1/\sqrt{2}$,

$$\pi_n = \frac{2a_{n+1}^2}{1 - \sum_{k=0}^{n} 2^k c_k^2} \text{ where } c_n = \frac{a_{n-1} - b_{n-1}}{2}\ .$$

In the limit,

$$\pi = \frac{2M^2(1, 1/\sqrt{2})}{1 - \sum_{k=0}^{\infty} 2^k c_k^2}\ .$$

After only eight iterations, π is accurate to 344 digits; 27 iterations gives over 100,000,000 digits!

[1]In this same year the symbol π was first proposed by an obscure English writer, William Jones, (π is the first letter in the Greek word peimetros or periphery)."π" became standard when Leonard Euler used it in "Introduction in Analsin Infinitron," 1748 .

[2]This is the way that UBASIC calculates π. These calculations were first done by hand, then with desk calculators, and finally with computers - these three periods are shown in the third, fourth, and fifth divisions of Table 5.

[3]The exception is Gosper's calculation of π using the continued fraction expansion.

For a more detailed and rigorous discussion of the AGM and other explicit algorithms for the calculation of π, see [1]. At the time this article was written, the record for finding the decimal expansion of π was over 206-billion decimal places, held by Takahasi and Kanada.[4]

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[4]It might be interesting to some readers that the current record for reciting π from memory is 42000 digits.