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UNIQUE-PERIOD PRIMES

Chris K. Caldwell
5856 Harry Daniels Road
Rives, Tennessee 38253
email: caldwell@utmartin.edu

Harvey Dubner
449 Beverly Road
Ridgewood, New Jersey 07450
email: 70372,1170@compuserve.com

Introduction

The reciprocal of every prime *p* (other than two and five) has a period, that is the decimal expansion of 1/*p* repeats in blocks of some set length. This period is the period of *p*. For example,

$$1/11 = .\overline{09}, \qquad 1/7 = .\overline{142857}, \qquad \text{and} \qquad 1/13 = .\overline{076923},$$

showing that the period of 11 is two, and that the periods of 7 and 13 are both six. Samuel Yates defined a unique-prime (or unique-period prime, see references [1,2 and 3]) to be a prime which has a period that it shares with no other prime. For example: 3, 11, 37 and 101 are the only primes with periods one, two, three and four respectively–so they are unique-primes. But 41 and 271 both have period five, 7 and 13 both have period six, 239 and 4649 both have period seven, and each of 353, 449, 641, 1409 and 69857 each have period thirty-two, showing that these primes are not unique-primes.

As we would expect from any object labeled "unique", unique-primes are extremely rare. For example, even though there are over 10^{47} primes below 10^{50}, only eighteen of these primes are unique-primes! These eighteen primes are listed in table one.

Insert Table 1 Near Here

In a letter dated 2/16/91, Samuel Yates announced the discovery of the twenty-ninth unique prime. In this article we show where to look for new unique primes, then give the results of our recent searches. The results include finding over a dozen new unique-primes, and the discovery of unique-primes with record lengths and record periods. We end by suggesting several possible avenues for further research.

How do You Find Unique Primes?

First, if a prime p has period n (so p is not 2 or 5), then

$$\frac{10^m}{p} - \frac{1}{p} = \frac{(10^m-1)}{p}$$

is an integer if and only if the decimal parts cancel out, that is, if and only if n divides m. So if p has period n, then p divides 10^m-1 for all m divisible by n, and in particular, the period of p is the smallest integer n for which p is a factor of 10^n-1. For example, since $10^1-1 = 3^2$, the only prime with period one is three. Next, it helps in our search to know we don't have to try all exponents because Fermat's (little) theorem states that p divides $10^{p-1}-1$, so it follows that the period of p must divide $p-1$. Finally, recall that the repunits R_n are written as n ones, that is, $R_n = (10^n-1)/9$. Obviously these repunits are closely related to the unique-primes. In fact, we can pull all these facts together as follows: **(i) The reciprocals of the primes two and five have terminating expansions, so two and five have no period. (ii) The only prime with period one is three. (iii) If p is a prime greater than five, then the period of p is the least n dividing $p-1$ such that p divides R_n.** For example, let p=23. Since $p-1 = 22$, we know the period of 23 is either 2, 11 or 22. A quick check shows 23 does not divide R_2(=11) or R_{11}, so 23 has period 22. (Primes p with period $p-1$ are often called full period primes.)

Back to the original question (how do you find unique-primes?): If we take R_m and divide out any factor it has in common with R_n (for each n that divides m), then what is left over is the product of the primes with period exactly m. If this quotient is the power of a single prime–then it is a unique-prime. We can make this more precise by introducing the cyclotomic polynomials $\Phi_n(x)$:

$$\Phi_n(x) = \prod_{d|n} (x^d-1)^{\mu(n/d)}, \qquad \text{so} \qquad 9R_n = 10^n - 1 = \prod_{d|n} \Phi_d(10).$$

where μ is the Möbius μ-function[1]. For example: $\Phi_1 = x-1$, $\Phi_2 = x+1$, $\Phi_3 = x^2+x+1$ and $\Phi_4 = x^2+1$. These polynomials are usually used as the first step in factoring any number of the

[1]Let m be a positive integer, then $\mu(m) = 0$ if m has a square factor; $\mu(m) = 1$ if m is a product of an even number of distinct primes (or m is one); and $\mu(m) = -1$ otherwise (that is, if m is the product of an odd number of distinct primes). For example: $\mu(1) = 1$, $\mu(2) = \mu(3) = -1$, $\mu(4) = 0$, $\mu(5) = -1$, and $\mu(6) = 1$.

form $b^n \pm 1$, see [4, section IIIC1]. Finally, the integers $\Phi_n(10)$ and $\Phi_{nm}(10)$ may not be relatively prime, but any factor that they have in common must divide nm. Now we have shown why: **(iv) *p* is a unique-prime of period *n* if and only if** $\mathbf{Q_n = \dfrac{\Phi_n(10)}{\gcd(\Phi_n(10),n)}}$ **is a power of *p*.**

Other than for p=3 (with period n=1 and $Q_1 = 3^2$), we have found no case where this quotient Q_n is a true power (a power greater than one). In fact, there is good reason to believe that $n = 1$ is the only case in which this happens.

The Repunit Connection

As indicated above, the first connection between repunits and unique-primes is through factorization. Most factorization tables of 10^n-1 and R_n are organized in terms of $\Phi_n(10)$, making unique-primes are easy to spot: just ignore factors of R_n that divide n (so what's left is Q_n) and look for the values of n with a single prime left. As a by product of our extending Yates' repunit factorization tables [5] we discovered twenty-three new unique probable-primes (probable-primes are positive integers p which satisfy Fermat's congruence $a^p \equiv 1 \pmod{p}$), see [4]).

Unfortunately, finding large probable-primes N is easy, but proving these large numbers are prime is usually not. For a large N, the primality tests based on partial factorizations of N+1 and N–1 are by far the easiest and fastest (see reference [4, 6 or 7]), so it is very fortunate that $\Phi_n(10)-1$ is often divisible by $\Phi_m(10)$ (for certain other m's with $m < n$), making these tests easier than usual. Because of this we were able to complete the primality proofs for fifteen new unique-primes (for more mathematical details see [8]). These new primes (and probable primes) are each marked with an asterisk in Table 2.

Insert Table 2 Near Here

A second and more direct connection between unique-primes and repunits is that when a repunit is prime, then it is unique! This is because (in this special case only) $R_n = \Phi_n(10)$. Currently only five repunit primes are known: R_2, R_{19}, R_{23}, R_{317}, and R_{1031}. The second author has used his special processors[2] to search up through n=16518, see [9] and [10]. Two facts the reader might find useful if they'd like to extend these search limits are (1) for R_n to be prime, n must be prime, and (2) any divisor of R_p (p and odd prime) has the form $2kp+1$ (for some integer k).

[2]A custom built processor for number theory which, for this type of calculation, is over 70 times as fast as a PC based on the Intel 486 processor. For information on an early version see [10]. These are now available as add-in boards for many IBM compatibles. Contact the second author for further information.

What is Next?

Other than showing where to search for new unique-primes, we have not even addressed the most basic of questions about these primes: are there infinitely many of them? Are there infinitely many repunit primes? Are there infinitely n such that $\Phi_n(10)$ is a power (greater than one) of a prime? We join others in conjecturing that the answers are yes, yes and no; but we are unable to prove any of these conjectures.

Above we just looked at the unique-primes in base ten. Other than the fact we have ten fingers, what is so special about ten? If we were switch to any other base, then the set of unique-primes would be different. For example, in base two (binary), all of the Mersenne Primes are repunit primes, so all Mersenne Primes are unique-primes base two, but what other primes are unique base two? More generally, are there infinitely many unique-primes base *b* for every base *b*? Are there infinitely many repunit primes base *b* for every base *b*? Are there infinitely occasions that $\Phi_n(b)$ is a power (greater than one) of a prime? (In binary such a prime would be a Wieferich prime.) Again, it seems the answers to these last three questions should be yes, yes and no, but...

Another direction that the reader might wish to pursue would be to generalize the notion of unique. For example, define bi-unique primes to be pairs of primes which have a period shared by no other primes. Since the average number of factors of $\Phi_n(10)$ should go to infinity (as n does), these should also become very rare (as should tri-unique...). See table three (an updated extension of Yates' table in [3]).

We would be pleased to hear of the readers' discoveries in any of these areas.

Insert Table 3 Near Here

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Table 1. The Seventeen Unique Primes less than 10^{50}

period	prime
1	3
2	11
3	37
4	101
10	9091
12	9901
9	333667
14	909091
24	99990001
36	9999990000 01
48	9999999900 000001
38	9090909090 90909091
19	1111111111 111111111
23	1111111111 1111111111 111
39	9009009009 0099099099 0991
62	9090909090 9090909090 9090909091
120	1000099999 9989998999 9000000010 001
150	1000009999 9999989999 8999990000 0000010000 1

Table 2.
The Known Unique-Prime and [Probable-Prime] Periods and Lengths

#	period n	digits	#	period n	digits	#	period n	digits
1	1	1	19	120	33	37*	1452	441
2	2	2	20	134	66	38*	1521	936
3	3	2	21	150	41	39*	1752	577
4	4	3	22	196	84	40*	1812	601
5	9	6	23	294	84	41*	1836	577
6	10	4	24	317	317	42*	1844	920
7	12	4	25	320	128	43*	[1862	757]
8	14	6	26*	385	241	44*	[2134	961]
9	19	19	27	586	292	45*	2232	721
10	23	23	28	597	396	46*	2264	1128
11	24	8	29	654	217	47*	[2667	1513]
12	36	12	30*	738	241	48*	3750	1001
13	38	18	31*	945	433	49*	[3903	2600]
14	39	24	32	1031	1031	50*	[3927	1920]
15	48	16	33*	1172	584	51*	[4274	2136]
16	62	30	34*	1282	640	52*	[4354	1861]
17	93	60	35*	1404	433			
18	106	52	36*	[1426	661]			

* New.

Table 3.
Number of Primes with Period Lengths in Given Range

Period Range	Exactly One	Exactly Two	Probably Two	Three or More	Not Yet Known
1-100	17	32	0	51	0
101-201	5	16	0	77	1
201-300	1	5	0	87	5
301-400	3	4	0	84	8
401-500	0	3	2	83	12
501-600	2	4	0	75	19
601-700	1	4	1	73	21
Totals	29	68	3	530	66