Math Colloquium
Five Lectures on Probability Theory
Part 1: Basics of Probability and Gambler's Ruin

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Probability Theory $=$ Mathematical framework for modeling/studying non-deterministic behavior where a source of randomness is introduced (this means that more than one outcome is possible)
The space of all possible outcomes is called the **sample space**. A set of outcomes is called an **event** and the source of randomness is called a **random variable**

Example (Single roll of a die)

Suppose we want to model the "experiment" of rolling a single unbiased die. The possible outcomes are the numbers $1, 2, 3, 4, 5, 6$. Hence the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. The random variable, call it X, represents the number we roll. Since the die is fair, we expect the chances of rolling any of the numbers $1, \ldots, 6$ to be equal. Hence the probability of each outcome has to be $1/6$. We have

$$P(X = 1) = P(X = 2) = \ldots = P(X = 5) = P(X = 6) = \frac{1}{6}.$$

Discrete Probability

A **discrete probability** space consists of a finite (or countable) set Ω of **outcomes** ω together with a set of non-negative real numbers p_ω assigned to each ω; p_ω is called the **probability of the outcome** ω. We require

$$\sum_{\omega \in \Omega} p_\omega = 1.$$

An **event** is a set of outcomes, i.e., a subset $A \subset \Omega$. The probability of an event A is

$$P(A) = \sum_{\omega \in A} p_\omega.$$

A **random variable** is a function X mapping the set Ω to the set of real numbers. We write $X\colon \Omega \to \mathbb{R}$. We note that if we have two disjoint events A and B (meaning that they do not share any outcomes), then

$$P(A \cup B) = P(A) + P(B).$$

Example (Rolling a Die Twice)

When rolling a fair die we introduced the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and the random variable $X\colon \Omega \to \{1, 2, 3, 4, 5, 6\}$ given by $X(i) = i$.
Suppose we wish to roll the die again, and obtain a second random variable Y that represents the number we get in the second roll. Our sample space Ω is inadequate. To accommodate the **new source of randomness** we need to extend the sample space to $\Omega \times \Omega = \{(i, j)\colon i, j = 1, 2, 3, 4, 5, 6\}$ with probability of each outcome $p_{ij} = 1/36$.
The random variable X is now modeled by a new function $X'\colon \Omega \times \Omega \to \{1, 2, 3, 4, 5, 6\}$ given by $X'(i, j) = i$. Moreover the random variable Y is modeled by $Y'\colon \Omega \times \Omega \to \{1, 2, 3, 4, 5, 6\}$ defined by $Y'(i, j) = j$. **The good news is that** for any $k = 1, 2, 3, 4, 5, 6$ we have

$$P(X = k) = P(X' = k),$$

since $\{X = k\} = \{k\} \subset \Omega$ and
$\{X' = i\} = \{(k, 1), (k, 2), (k, 3), (k, 4), (k, 5), (k, 6)\} \subset \Omega \times \Omega$.

Fundamental Assumption of Probability Theory

To have freedom of introducing new sources of randomness we require that:
Probability Theory only studies concepts and operations that are preserved under extension of underlying sample space.
For that reason the sample space often disappears form our view.

Example (Rolling a Die Twice Continued)

Suppose we roll a die twice and consider the random variable $X + Y$ represented by the function $X' + Y'$ that gives the sum of numbers we rolled. Then

$$X' + Y' \colon \Omega \times \Omega \to \{2, 3, \dots, 11, 12\}.$$

We compute

$$P(X' + Y' = 2) = P(\{(1, 1)\}) = 1/36$$
$$P(X' + Y' = 3) = P(\{(1, 2), (2, 1)\} = 2 \cdot (1/36) = 1/18.$$

Example (Rolling a Die Twice Continued)

We continue our compuation of the probabilities for the random variable $X' + Y'$ and obtain

k	2	3	4	5	6	7	8	9	10	11	12
p_k	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

We have obtained a new probability space

$$\Omega' = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

with probability p_k of each outcome given in the table above. This probability is called the **distribution** of the random variable $X' + Y'$ (or $X + Y$).

Often the information provided by the probability distribution is all we need to know about the random variable. This is yet another reason for the sample space disappearing from our view.

Conditional Probability

Let Ω be a sample space with probability P. This means that to an event A (here $A \subset \Omega$) we have assigned the probability $P(A)$ of A happening. Suppose we have two events A and B with $P(B) > 0$. The **conditional probability** of A given B is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example

A pair of fair dice is rolled. What is the conditional probability that one of the dice is four given the sum is seven? Denote
$A = \text{one of the dice is four}$ and $B = \text{the sum is seven}$. Then

$$B = \{(1,6),(6,1),(2,5),(5,2),(4,3),(3,4)\} \implies P(B) = 6/36 = 1/6$$
$$A \cap B = \{(4,3),(3,4)\} \implies P(A \cap B) = 2/36 = 1/18$$

Therefore $P(A|B) = (1/18)/(1/6) = 1/3$.

Conditional Probability Continued

- For two events A and B we have

$$P(A \cap B) = P(A|B)P(B)$$

- **The Law of Total Probability**: For two events A and C

$$\begin{aligned} P(A) &= P(A \cap C) + P(A \cap C^c) \\ &= P(A|C)P(C) + P(A|C^c)P(C) \end{aligned}$$

  In particular for conditional probability we have

$$P(A|B) = P(A \cap C|B) + P(A \cap C^c|B).$$

- For three events A, B, C we have

$$P(A \cap B|C) = P(A|B \cap C)P(B|C).$$

Proof of $P(A \cap B|C) = P(A|B \cap C)P(B|C)$

We compute

$$\begin{aligned} P(A|B \cap C)P(B|C) &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)} \\ &= \frac{P((A \cap B) \cap C)}{P(C)} \\ &= P(A \cap B|C). \end{aligned}$$

Independence

Intuitively, two events are **independent** if one of them happening has no influence on the other happening. If we denote the events by A and B we need

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

Therefore

$$\frac{P(A \cap B)}{P(B)} = P(A) \quad \text{and} \quad \frac{P(B \cap A)}{P(A)} = P(B)$$

We obtain that $P(A \cap B) = P(A)P(B)$.
Two events A and B are called **independent** if $P(A \cap B) = P(A) \cdot P(B)$.
Two random variables X and Y defined on some discrete probability space are called **independent** if for any two real numbers i and j subsets $A, B \subset \mathbb{R}$

$$P(X = i, Y = j) = P(X = i)P(Y = j)$$

Independence - Examples

Example (Dependent Events)

We roll a die. Consider the events

$$A = \text{the number on the die is less than } 4$$
$$B = \text{the number on the die is } 3, 4 \text{ or } 5$$

Then $P(A) = 1/2$, $P(B) = 1/2$, and $P(A \cap B) = P(\{3\}) = 1/6$.

Example (Independent Events)

Toss two fair coins. The sample space is
$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$. Consider the events

$$A = \text{the first coin lands on heads} = \{(H, H), (H, T)\}$$
$$B = \text{the second coin lands on heads} = \{(H, H), (T, H)\}$$

Then $P(A) = 1/2$, $P(B) = 1/2$, and $P(A \cap B) = P(\{(H, H)\}) = 1/4$.

Gambler's Ruin

You, the Gambler, are playing independent repetitions of a fair game against the House. If you win, you gain a dollar from the House and when you lose, you lose a dollar to the House. Suppose you start with n dollars and the house starts with m dollars.
What is the probability the House is ruined before you are?
For $j = 0, 1, \ldots, n+m$ we define

$$p_j = \text{probability the we win provided we start with } j \text{ dollars}$$

Then

$$p_0 = 0 \quad \text{and} \quad p_{n+m} = 1.$$

We need to find p_n.
The goal is to get a relation between the numbers p_j with the above being the boundary condition. We will obtain a discrete **Boundary Value Problem**.

Gambler's Ruin Continued

We introduce the following events

$$\begin{aligned} H &= \text{the House is ruined} \\ W &= \text{we win a dollar} \\ L &= \text{we lose a dollar} \end{aligned}$$

Then we have

$$\begin{aligned} p_j &= P(H| \text{ start with } \$j) \\ &= P(H \cap W| \text{ start with } \$j) + P(H \cap L| \text{ start with } \$j) \\ &= P(H|W \cap \text{ start with } \$j)P(W| \text{ start with } \$j) \\ &+ P(H|L \cap \text{ start with } \$j)P(L| \text{ start with } \$j) \\ &= \frac{1}{2}p_{j+1} + \frac{1}{2}p_{j-1} \end{aligned}$$

Gambler's Ruin Continued

We have obtained a relation

$$p_j = \frac{1}{2}p_{j+1} + \frac{1}{2}p_{j-1}, \quad j = 1, 2, \ldots, n+m-1$$

with boundary conditions

$$p_0 = 0 \quad \text{and} \quad p_{n+m} = 1.$$

To solve for p_j we write

$$\frac{1}{2}p_j + \frac{1}{2}p_j = \frac{1}{2}p_{j+1} + \frac{1}{2}p_{j-1}$$

Therefore

$$p_j + p_j = p_{j+1} - p_{j-1},$$

hence

$$p_{j+1} - p_j = p_j - p_{j-1} \quad \text{for } j = 1, 2, \ldots, n+m-1$$

Gambler's Ruin Continued

Using the last relation we have for a fixed j (here $j = 1, 2, \ldots, n+m$)

$$\begin{aligned} p_j - p_{j-1} &= p_{j-1} - p_{j-2} \\ &= p_{j-2} - p_{j-3} = \\ &= \ldots \\ &= p_1 - p_0 \\ &= p_1 \end{aligned}$$

Therefore for

$$\begin{aligned} p_j &= (p_j - p_{j-1}) + (p_{j-1} - p_{j-2}) + \ldots + (p_1 - p_0) \\ &= jp_1. \end{aligned}$$

It remains to find p_1. Use use the second boundary condition that $p_{n+m} = 1$. Since $p_{n+m} = (n+m)p_1$, we get

$$p_1 = \frac{1}{n+m}$$

Gambler's Ruin Continued

We have obtained

$$p_j = \frac{j}{n+m} \quad \Longrightarrow \quad p_n = \frac{n}{n+m}.$$

We note that when $n = m$, i.e., the House and the Gambler start with the same amount of money, the probability of wining is $1/2$.

Example

If we play an unfavorable game with probability of winning $P(W) = p$, then

$$p_n = \frac{1 - \left(\frac{1-p}{p}\right)^n}{1 - \left(\frac{1-p}{p}\right)^{n+m}}$$

For example if the probability of winning is $p = 49/100$ with $n = 1000$ and $m = 100$, then the probability of winning \$100 from the house is $0.0183 = 1.83\%$.

References

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