Math Colloquium
Five Lectures on Probability Theory
Part 5: Brownian Motion

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Brownian Motion

Brownian Motion = Motion of random fluctuations of pollen immersed in water observed by *Robert Brown* in 1827.

Mathematical Model suggested/formulated by Norbert Wiener in 1923 as a **stochastic process** .

A continuous time **stochastic process** is a collection $(X_t)_{t\geq 0}$ of random variables $X_t\colon \Omega \to S$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with values in a measurable space $(S, \mathcal{S})$, called the **state space**.

Let $(X_t)_{t\geq 0}$ be a stochastic process. Then for and $\omega \in \Omega$ the mapping

$$[0, \infty) \ni t \mapsto X_t(\omega) \in S$$

is called a **sample path/trajectory** of ω. We think of ω as a certain realization of the possible scenarios.

Precise Definition of Brownian Motion

A 1-dimensional **Brownian Motion** starting at 0 is a stochastic process $(B_t)_{t\geq 0}$ with values in real numbers satisfying

- $B_0 = 0$ almost surely
- the process has **independent increments**, which means that for any times
$$0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$$
the random variables
$$B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$
are independent.
- For any $t \leq 0$ and $h > 0$ the increment $B_{t+h} - B_t$ is normally distributed with expectation 0 and variance h. This means that
$$P(a \leq B_{t+h} - B_t \leq b) = \frac{1}{\sqrt{2\pi h}} \int_a^b e^{-\frac{x^2}{2h}}\, dx$$
- For almost every $\omega \in \Omega$ the sample path $t \mapsto B_t(\omega)$ is continuous.

Scaling Properties of of Brownian Motion

Theorem (Wiener, 1923)

Brownian Motion exists.

Let $(B_t)_{t\geq 0}$ be a Brownian Motion. Then

- For any $s \geq 0$ the process $(B_{t+s} - B_s)_{t\geq 0}$ is a Brownian Motion.
- For any $\lambda > 0$ the process $(\lambda B_{\lambda^2 t})_{t\geq 0}$ is a Brownian Motion.
- The process $(W_t)_{t\geq 0}$ defined by

$$W_t = \begin{cases} 0 & t = 0 \\ tB_{1/t} & t > 0 \end{cases}$$

  is a Brownian Motion.

Donsker's Invariance Principle

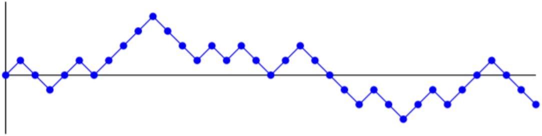
Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent identically distributed random variables with Bernoulli distribution, i.e.,

$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

Then the sequence of partial sums

$$S_n = \sum_{k=1}^{n} X_k$$

is the **simple random walk**, which we visualize by the graph (n, S_n).
Here: **jump increment/scale**$= 1$, **time increment/scale**$= 1$.



Idea: Reduce the time and jump increments appropriately and take the limit when the size of the increments goes to zero.

Let $h > 0$ be the jump increment and h^2 be the time increment. Define

$$Z_t^h = hS_{t/h^2} \quad \text{for} \quad t/h^2 \in \mathbb{N}$$

and extend to all values of t by piece-wise linearity. More precisely, define for any $t \geq 0$

$$Z_t^h = h\left(1 - \left\{\frac{t}{h^2}\right\}\right) S_{[t/h^2]} + h\left\{\frac{t}{h^2}\right\} S_{[t/h^2]+1},$$

where $[x]$ denotes the integer part of x and $\{x\} = x - [x]$ is the non-integer part of x.

Theorem (Donsker)

The process (Z_t^h) converges in distribution when $h \to 0$ to Brownian Motion.

We note that

$$Z_t^h = hS_{[t/h^2]} + h\left\{\frac{t}{h^2}\right\} X_{[t/h^2]+1}$$

Hence

$$E[Z_t^h] = 0 \quad \text{and} \quad \mathrm{Var}[Z_t^h] = h^2\left[\frac{t}{h^2}\right] + h^2\left\{\frac{t}{h^2}\right\}^2.$$

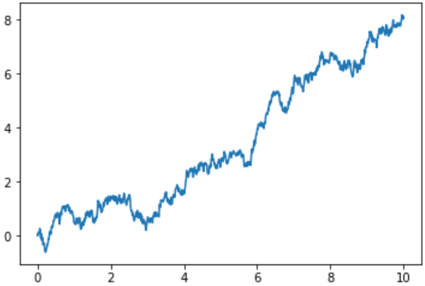
Moreover, for two discrete times $s = mh^2$ and $t = nh^2$ with $m < n$ we have

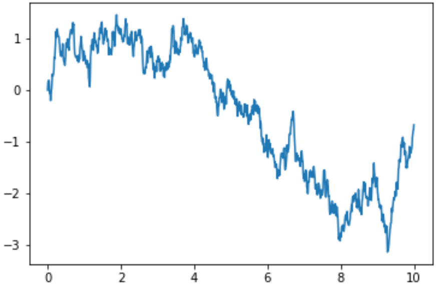
$$Z_t^h - Z_s^h = h\sum_{k=m+1}^{n} X_k \to N(0, t-s),$$

by Central Limit Theorem, where the convergence is in distribution, since

$$\mathrm{Var}\left[h\sum_{k=m+1}^{n} X_k\right] = h^2(n-m) = t-s.$$

Sample paths





Kolmogorov's Construction of Brownian Motion

Step 1: **Caratheodory's Extension Theorem**
Let X be a set. An **outer measure** is a mapping $\mu\colon 2^X \to [0,\infty]$ satisfying the conditions that

- $\mu(\emptyset) = 0$,
- if $A \subset \cup_{n=1}^{\infty} A_n$, then $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

We say that a set $A \subset X$ is μ-**measurable** in the sense of Caratheodory if for any $E \subset X$

$$\mu(E) = \mu(E \cap A) + \mu(E - A). \tag{1}$$

Theorem: The collection of all μ-measurable sets is a σ-algebra, which we denote by $\mathfrak{m}(\mu)$, and $\mu\colon \mathfrak{m}(\mu) \to [0,\infty]$ is a measure such that if $\mu(A) = 0$, then $A \in \mathfrak{m}(\mu)$.

Step 2: Caratheodory's Construction of Measure

A collection $\mathfrak{R} \subset 2^X$ is called a **semi-ring** if

- $\emptyset \in \mathfrak{R}$,
- $A, B \in \mathfrak{R} \implies A \cap B \in \mathfrak{R}$,
- $A, B \in \mathfrak{R} \implies$ there exist disjoint sets $C_1, \ldots, C_n \in \mathfrak{R}$ such that $A - B = C_1 \cup \ldots \cup C_n$.

Theorem: Let $\mathfrak{R} \subset 2^X$ be a semi-ring and let $\mu \colon \mathfrak{R} \to [0, \infty]$ be a countably additive mapping, i.e.,

- $\mu(\emptyset) = 0$,
- if $A_n \in \mathfrak{R}$, $n \in \mathbb{N}$ are disjoint with $\cup_{n \in \mathbb{N}} A_n \in \mathfrak{R}$, then $\mu(\cup_{n \in \mathbb{N}} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

For any set $E \subset X$ define

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) \colon A_n \in \mathfrak{R} \text{ and } E \subset \cup_{n=1}^{\infty} A_n \right\} \qquad (2)$$

Then μ^* is an outer measure; the measure $\mu^*|_{\mathfrak{m}(\mu^*)}$ extends the countable additive mapping μ, meaning that for any $A \in \mathfrak{R}$ we have that $A \in \mathfrak{m}(\mu^*)$ and $\mu(A) = \mu^*(A)$; and we have $\sigma(\mathfrak{R}) \subset \mathfrak{m}(\mu^*)$.

Step 3: Kolmogorov's Consistency Theorem

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process with real values. For any finite sequence

$$T = \{0 \leq t_1 < t_2 < \ldots < t_n\}$$

define a Borel probability measure P^T on $\mathbb{R}^n$ as the distribution probability of the random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, i.e., for any Borel set $U \subset \mathbb{R}^n$

$$P^T(U) = P((X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \in U)$$

or for cylinders

$$P^T(A_1 \times A_2 \times \ldots \times A_n) = P(X_{t_1} \in A_1, X_{t_2} \in A_2, \ldots, X_{t_n} \in A_n)$$

where $A_1, A_2, \ldots, A_n$ are Borel subsets of $\mathbb{R}$.

The probabilities P^T, where T ranges over all finite sequences, are called **finite dimensional distributions** (FDDs) of the stochastic process $(X_t)_{t\geq 0}$.

The following **consistency condition** holds true. If $S \subset T$ are two finite sequences, then

$$\pi_S^T \sharp P^T = P^S,$$

where for $\pi_S^T \colon \mathbb{R}^{|T|} \to \mathbb{R}^{|S|}$ is the obvious projection.

For example for $S = \{t_1 < t_3\} \subset \{t_1 < t_2 < t_3\}$ we have

$$P^T(A_1 \times \mathbb{R} \times A_3) = P^S(A_1 \times A_3).$$

Goal: We will reverse the process. Having a family of probability measure $(P^T)_T$, we will construct a stochastic process whose finite dimensional distributions are exactly the probability measures P^T. This construction is called **Kolmogorov's Consistency Theorem**.

Theorem: Let $(P_T)_T$ be a collection of Borel probability measures, with each measure P_T defined on the Euclidean space $\mathbb{R}^{|T|}$, satisfying the consistence assumption

$$S \subset T \implies \pi^T_S \sharp P^T = P^S.$$

Then there exists a unique probability measure P on

$$\Omega = \{\omega \colon [0, \infty) \to \mathbb{R}\} = \{(\omega_t)_{t \geq 0}\}$$

such that

$$\pi^\Omega_T \sharp P = P^T,$$

where $\pi^\Omega_T \colon \Omega \to \mathbb{R}^{|T|}$ is the projection

$$\pi^\Omega_T((\omega_t)_{t\geq 0}) = (\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n})$$

Moreover the function $X_\cdot(\cdot) \colon [0, \infty) \times \Omega \to \mathbb{R}$ given by

$$X_t(\omega) = \omega(t)$$

defines a stochastic process with finite dimensional distributions P^T.

Sketch of Proof: Let $\mathfrak{R}$ be the collection of finite dimensional cylinders, i.e., sets of the form

$$\left(\pi^{\Omega}_{T}\right)^{-1}(A_1 \times A_2 \times \ldots \times A_n),$$

where $T = \{0 \leq t_1 < t_2 < \ldots < t_n$ and $A_i \subset \mathbb{R}$, $i = 1, \ldots, n$, are Borel sets. The $\mathfrak{R}$ is a semi-ring (actually an algebra).

We define a set function $P_0 \colon \mathfrak{R} \to [0,1]$ by

$$P_0\left(\left(\pi^{\Omega}_{T}\right)^{-1}(A_1 \times A_2 \times \ldots \times A_n)\right) = P^T(A_1 \times A_2 \times \ldots \times A_n).$$

Then P_0 is well-defined by consistency assumption and satisfies the assumptions of Caratheodory Construction of Measure Theorem, hence it admits an extension to a probability measure on the σ-algebra $\sigma(\mathfrak{R})$ generated by $\mathfrak{R}$.

Step 4: **Choosing the correct collection of FDDs**: We will define a collection $(P^T)_T$ of finite dimensional distributions that will give us Brownian Motion via Kolmogorov's consistency theorem.

For $T = \{0 \leq t_1 < t_2 < \ldots < t_n\}$ and for a Borel set $U \subset \mathbb{R}^n$ define

$$P^T(U) = \int_U \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left[-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right] \right) \, dx_1 \cdots dx_n,$$

where $t_0 = x_0 = 0$.

Then the collection $(P^T)_T$ satisfies the consistency condition. Hence we get a Stochastic Process, which we denote $(B_t)_{t \geq 0}$.

The stochastic process $(B_t)_{t \geq 0}$ satisfies the first three defining conditions of Brownian Motion.

For example, for $t \geq 0, h > 0$ and $b \in \mathbb{R}$ we have

$$P(B_{t+h} - B_t \leq b) = P(\omega \in \Omega \colon (\omega_t, \omega_{t+h}) \in U),$$

where $U = \{(x, y) \colon y - x \leq b\}$. We write that $P(B_{t+h} - B_t \leq b)$ equals

$$\frac{1}{\sqrt{(2\pi)^2 th}} \int_U \exp\left[-\frac{1}{2}\left(\frac{(y-x)^2}{h} + \frac{x^2}{t}\right)\right] dy\, dx$$

We substitute $u = x$ and $v = (y - x)$. Then $dx \wedge dy = du \wedge dv$, hence the above integral takes the form

$$\frac{1}{\sqrt{(2\pi)^2 th}} \int_{-\infty}^{\infty} \int_{-\infty}^{b} \exp\left[-\frac{1}{2}\left(\frac{u^2}{t} + \frac{v^2}{h}\right)\right] dv\, du$$
$$= \frac{1}{\sqrt{2\pi h}} \int_{-\infty}^{b} \exp\left[-\frac{1}{2}\left(\frac{v^2}{h}\right)\right] dv$$

Step 5: **Kolmogorov Continuity Theorem**: We need to modify the process $(B_t)_{t\geq 0}$ to make sure the trajectories are continuous. This is possible due to the following:

Theorem: Let $(X_t)_{t\geq 0}$ be a real valued process on a probability space $(\Omega, \mathfrak{F}, P)$. If for any $T > 0$ there exist constants $\alpha > 0$, $\beta > 0$ and $K > 0$ such that

$$E[|X_t - X_s|^\alpha] \leq K|t-s|^{1+\beta} \quad \text{for all } s, t \in [0, T],$$

then there exists a modification $\tilde{X}$ of X with almost all trajectories continuous. More precisely,

- for almost every $\omega \in \Omega$ the trajectory $t \mapsto \tilde{X}_t(\omega)$ is continuous.
- for every time t we have that $P(X_t = \tilde{X}_t) = 1$.

Moreover, the trajectories of $(\tilde{X})_{t\geq 0}$ are locally γ-Holder continuous with $0 < \gamma < \frac{\beta}{\alpha}$.

For the Brownian Motion $(B_t)_{t \geq 0}$ one can show that $\alpha = 4$, $\beta = 1$ and $K = 3$ do the job. Namely,

$$E[|B_t - B_s|^4] = 3|t - s|^2,$$

since $B_T - B_s$ has normal distribution with mean 0 and variance $|t - s|$. and $E[|B_t - B_s|^4]$ is called the fourth absolute moment of $B_T - B_s$.

Theorem (Paley,Wiener, Zygmund,1933)

Almost surly the trajectories are nowhere differentiable.

References

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