Space Filling Curves

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Abstract

We present a geometric construction of a space filling curve, i.e., a continuous function mapping the closed interval $[0,1]$ **onto** the closed unit square $[0,1] \times [0,1]$ in the plane. We follow the original idea of David Hilbert from the end of the nineteenth century.

Introduction

A *plane curve* is a continuous mapping γ of the closed interval $[0,1]$ into the Euclidean plane $\mathbb{R}^2$. We think of γ as a trajectory of an object traveling from the *starting point* $\gamma(0)$ to the *ending point* $\gamma(1)$.

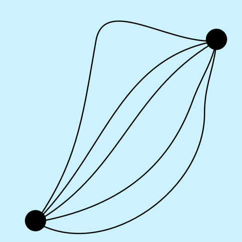
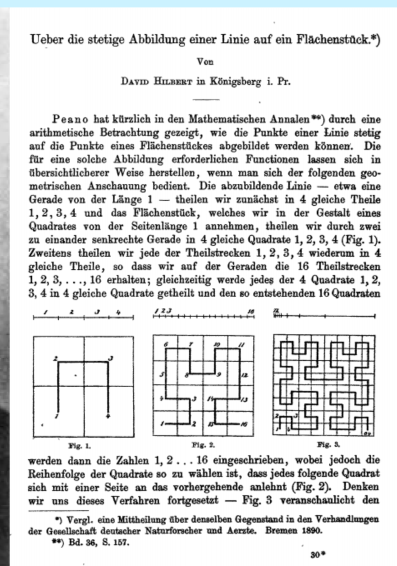


Figure: Five different curves connecting points A and B

A *space filling curve* is a plane curve that passes through every point of the unit square $[0,1] \times [0,1]$. Such a curve was first discovered by *Giuseppe Peano* (see [2]) in 1890 , which was a shocking result at that time. Just one year later *David Hilbert* (see [1]) simplified the idea of Peano giving the first geometric construction of a space filling curve. We present this idea here.
Below, left to right, we see Giuseppe Peano, David Hilbert, and the first page of the original paper of Hilbert with the construction of his space filling curve.





The Construction

We construct a sequence (γ_n) of plane curves, which we will prove converges to a space filling curve. The elements γ_n of the sequence are called the *iterations* of the Hilbert curve. The construction of the sequence is inductive. This means we need to (1) define the first curve γ_1 and (2) knowing an n-th curve γ_n, we need to show how to construct γ_{n+1}, the $(n+1)$-st one. For clarity we define γ_1 and show how to obtain γ_2 and γ_3. The inductive step $\gamma_n \mapsto \gamma_{n+1}$ will then be clear.

The First Curve γ_1

The curve γ_1 is a curve consisting of three line segments connecting the points $(\frac{1}{4},\frac{1}{4})$, $(\frac{1}{4},\frac{3}{4})$, $(\frac{3}{4},\frac{3}{4})$, and $(\frac{3}{4},\frac{1}{4})$. We call these points *vertices*. Hence we have 4 vertices.

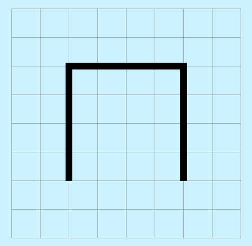


Figure: The first iteration of the Hilbert curve

The Second Curve γ_2

We divide the square $[0,1] \times [0,1]$ into four identical squares: *Square 1*: $[0,\frac{1}{2}] \times [0,\frac{1}{2}]$, *Square 2*: $[0,\frac{1}{2}] \times [\frac{1}{2},1]$, *Square 3*: $[\frac{1}{2},1] \times [\frac{1}{2},1]$, and *Square 4*: $[\frac{1}{2},1] \times [0,\frac{1}{2}]$. In each square we put the first curve γ_1 scaled by the factor of $1/2$ rotated appropriately. In the first square γ_1 is rotated 90° clockwise, in the second and third squares γ_1 stays in its original position, and in the fourth square it is rotated 90° counterclockwise. Next we connect the ending point of the curve in Square 1 with the staring point of the curve in Square 2. We repeat this process for curves in Squares 2 and 3, and Squares 3 and 4.

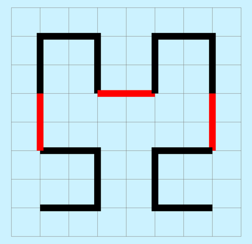


Figure: The second iteration of the Hilbert curve

The vertices of our second curve γ_2 are $(\frac{1}{8},\frac{1}{8})$, $(\frac{3}{8},\frac{1}{8})$, $(\frac{3}{8},\frac{3}{8})$, $(\frac{1}{8},\frac{3}{8})$, $(\frac{1}{8},\frac{5}{8})$, $(\frac{1}{8},\frac{7}{8})$, $(\frac{3}{8},\frac{7}{8})$, $(\frac{3}{8},\frac{5}{8})$, $(\frac{5}{8},\frac{5}{8})$, $(\frac{5}{8},\frac{7}{8})$, $(\frac{7}{8},\frac{7}{8})$, $(\frac{7}{8},\frac{5}{8})$, $(\frac{7}{8},\frac{3}{8})$, $(\frac{5}{8},\frac{3}{8})$, $(\frac{5}{8},\frac{1}{8})$, and $(\frac{7}{8},\frac{1}{8})$. There are $16 = 4^2$ of them and they can be written as $\frac{m}{2^3}$, where $m = 1,3,5,7$.

The Third Curve γ_3

The construction of γ_3 from γ_2 is the same as the construction of γ_2 from γ_1. We provide the graph of γ_3 and a description of its vertices.

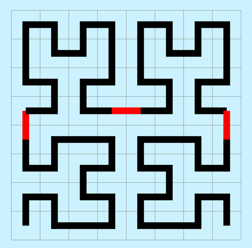


Figure: The third iterations of the Hilbert curve

There are $64 = 4^3$ vertices. They are of the form $\frac{m}{2^4}$, where m ranges over the numbers $1,3,5,7,9,11,13,15$, i.e., the odd numbers between 1 and $2^4 - 1$.

Sketch of Proof

We need to show that the sequence (γ_n) of plane curves we have constructed converges and that the limit is a space filling curve. We divide the proof into two sets providing only the main ideas.

Step 1. The sequence (γ_n) converges

This follows from the fact that the sequence (γ_n) is *Cauchy* with respect to *uniform convergence*. In particular, it converges and the limit is a continuous mapping, i.e., a plane curve. We denote the limit curve by γ.

Step 2. The curve γ is space filling

Remark

To fully understand the argument of this step we need some familiarity with the notions of closed, dense, and compact sets of the plane $\mathbb{R}^2$. They belong to a course on the theory of metric spaces. We refer to the monograph [3] for a study of these spaces.

The image of our curve γ is a compact, hence closed, subset of the square $[0,1] \times [0,1]$. Since the set of points $(\frac{m}{2^k},\frac{n}{2^k})$, where k is any positive integer, and m,n range over odd numbers between 1 and $2^k - 1$, is a dense subset of the square $[0,1] \times [0,1]$. By construction of the sequence (γ_n), the same can be said about the image of γ. Finally, the image of γ, as a closed and dense subset of the square $[0,1] \times [0,1]$, fills the entire square.

References

- [1] David Hilbert, Ueber die stetige Abbildung einer Linie auf ein Flachenstuck. Mathematischen Annalen 38, (1891), no. 3, 459-460.
- [2] Giuseppe Peano, Sur une courbe, qui remplit toute une aire plane. Mathematischen Annalen 36, (1890), no. 1, 157-160.
- [3] Walter Rudin, Principles of mathematical analysis. Third edition. International Series in Pure and Applied Mathematics. McGraw–Hill Book Co., New York-Auckland-Dusseldorf, 1976.